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Applied Mechanics. — *On a special case of bending.* By C. B. BIEZENO.

(Communicated at the meeting of April 25, 1942.)

1. *Introduction.* In a recent paper R. SONNTAG [1]) draws attention to a special case of bending, which occurs if a highly elastic beam, freely supported in two points of prescribed distance l is subjected to a transverse load P, acting in the middle of the span. The beam is supposed to slide freely — and without friction — over its supports, so that great deflections are to be expected even under relatively small loads. In an ingenious, but rather artificial way the author succeeds in deducing the relations between the load P, the corresponding deflection f, and the length of the deflected beam as far as it lies between its two supports. The artificiality of the method lies in the linearisation of the problem, which — at the expense of a rather laborious control — is proved to hold within well-mentioned limits.

In the following remarks it is shown that the problem admits a quite natural solution, which for all possible beams requires the construction of one single graph from which SONNTAG's conclusions — inclusive the interesting remark about the stability of the beam — can all be derived.

2. *The graphical method.* In this section the central line of the girder under consideration will be constructed by means of a method which with a slight modification is identical with the so-called method of elastic joints [2]). As to the notation it may be stated, that:

M stands for the bending moment in any section of the girder
EI for its flexural rigidity
$1/\varrho$ for the curvature of the central line
l_e for its effective length (the length between the points of support A and B)
l for the length of the span AB
f for the maximum deflection of the central line
P for the central load
N for the reactions of the supports
β for the slope of the central line at A and B
"a" for the length of an "element" of the girder
EI/a for the elastic stiffness of such an element.

Fig 1 represents the bent girder under the action of the load P, giving rise to the normal reactions $N = P/2 \cos\beta$ of the supports. If P is prescribed, it is required to determine N, β, l_e and f in terms of P and it is easily seen which way has to be followed if we endeavour a direct solution of this problem. However elliptic integrals will be involved in the calculations, which on account of the undeterminate length l_e will become rather troublesome. It therefore recommends itself to have rescue to an indirect method of attack by making the problem dependent on the normal reaction N. If an arbitrary value of N, say $N = 100$ kg be assumed, the central line of the girder can be constructed step by step, as illustrated in fig. 2.

Let the construction of the central line have been performed up to the endpoint T_{23} of the "element", $T_{12}T_{23}$ of length a, let S_2S_3 represent the tangent to the central line

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[1]) Comp. R. SONNTAG, Der beiderseitig gestützte, symmetrisch belastete gerade Stab mit endlicher Durchbiegung und seine Stabilität. Ingenieur-Archiv, Bnd. XII, S. 283—307.

[2]) Comp. f.i. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, III, 4, 19, p. 172.

in this endpoint and let S_3 be acquired by making $T_{23}S_3$ equal to $a/2$. Then with high approximation the mean value of the bending moment M_3 occurring in the next element $T_{23}T_{34}$ of the girder will be given by the value $M_3 = Nb_3$, and consequently the radius of curvature ϱ_3 of this element is represented by $EI : M_3 = EI : Nb_3$. The centre of curvature N_3 of this element therefore can be constructed. A circle with this point as centre and with radius ϱ_3 can be drawn, and the next point T_{34} of the central line,

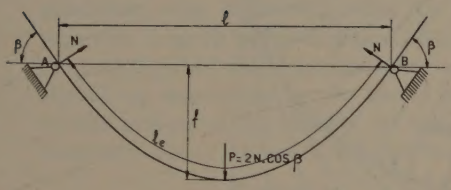

Fig. 1.

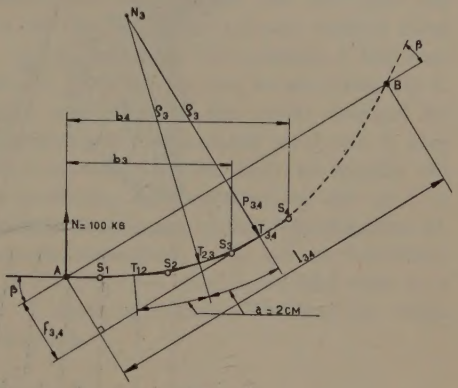

Fig. 2.

together with the corresponding tangent in this point is obtained by pacing the length $T_{23}T_{34}$ = "a" on the circle or by rotating the radius N_3T_{23} through the angle $\triangle \beta = Nba/EI$. By making $T_{34}S_4$ equal to $a/2$, and by reading off the distance b_4 the mean bending moment of the next element of the girder is found, a.s.o.

If for a moment the construction be stopped at T_{34} and if N_3T_{34} be made a line of symmetry for a curve one half of which is represented by $AT_{12}T_{23}T_{34}$, a central line is obtained, the span and deflection of which are represented by l_{34} and f_{34}. The load P_{34} necessary to maintain the prescribed deflection is fixed by the equation of equilibrium $P_{34} = 2N \cos \beta$. The construction hitherto described has been carried out in fig. 3 up to the point 15 for a girder, whose flexural rigidity amounts to 25000 kg cm^2. The elementary length "a" has been chosen equal to 2 cm; the normal reaction N in A has again been chosen 100 kg. We read from this construction that a deflection $f_{7,8}$ occurs if the span AB (fig. 1) is equal to $l_{7,8}$ and if the girder is centrally loaded by $P_{7,8}$ (see the upper left corner of fig. 3, where a half-circle has been drawn on a vertical diameter representing 200 kg, and where $P_{7,8}$ is parallel to the normal, passing through the points N_7 and N_8 of the main figure).

The accuracy of the construction can be checked by calculating the deflection f corresponding to that point of the central line for which the tangent is parallel to the normal n_1 of the point A. If a rectangular system of coordinates be assumed, the y-axis of which coincides with the normal n_1, the x-axis with the line $A\,1$, then the differential epuation of the central line is given by:

$$\frac{EI\,y''}{(1+y'^2)^{3/2}} = Nx. \quad \ldots\ldots\ldots \quad (1)$$

The first integral of this equation is found in an elementary way by putting $y' = tg\,t$. Taking into account that y' must be zero for $x = 0$, we find

$$y' = \frac{a\,x^2}{\sqrt{1-a^2\,x^4}} \quad \text{with} \quad a = \frac{N}{EI}. \quad \ldots\ldots \quad (2)$$

The derivative y' becomes infinite if $x = a^{-1/2}$, which in our case ($EI = 25000$ kg cm^2, $N = 100$ kg) leads to $x = 22{,}37$ cm. In figure 3 the point for which y' is infinite can

accurately be constructed. Its distance to the y-axis is (in the fullscale drawing) undistinguishable from the calculated value.

If we should have started our construction with another normal reaction say N^*, and if the flexural rigidity of the girder should have been EI^* instead of EI, another

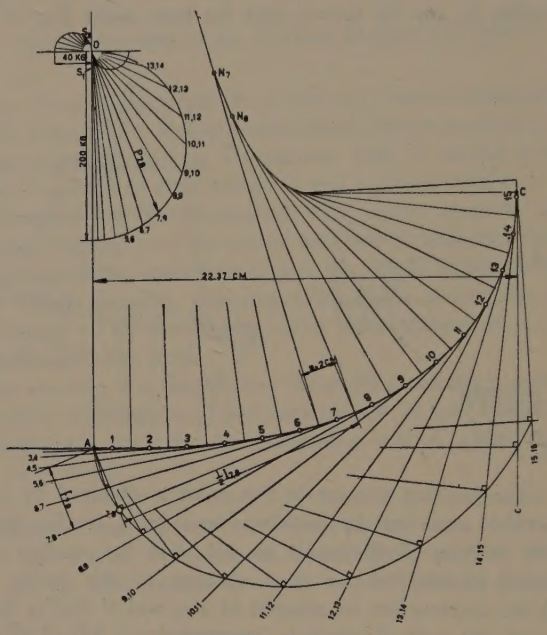

Fig. 3.

central line would have been obtained, provided the same length-scale should have been used. But it is easily understood that the central line of fig. 3 would serve our aim as well if only the scale to wich the figure has been designed, is suitably changed. Assuming that 1 cm of the figure represents p cm in reality, the quantity $\triangle \beta^* = N^*.pb.pa./EI^*$,

TABLE 1.

Girder-element	β	P	l_e	l	f	$\lambda = Pl^2/8\,EI$	f/l	l_e/l
1— 2	0.008	200.0	4	4	0.00	0.016	0	1
2— 3	0.032	200.0	8	8	0.10	0.064	0.0125	1
3— 4	0.0718	199.8	12	12	0.25	0.144	0.0208	1
4— 5	0.1276	198.7	16	16	0.47	0.254	0.0294	1
5— 6	0.1993	196.4	20	19.80	1.21	0.385	0.0612	1.01
6— 7	0.2866	192.1	24	23.54	2.20	0.533	0.0934	1.02
7— 8	0.3897	185.1	28	27	3.50	0.674	0.1296	1.037
8— 9	0.5079	174.8	32	30	5.20	0.786	0.1733	1.067
9—10	0.6404	160.1	36	32.40	7.30	0.840	0.2253	1.111
10—11	0.7855	141.3	40	33.86	9.80	0.809	0.2894	1.182
11—12	0.9416	117.9	44	34.40	12.45	0.697	0.3620	1.280
12—13	1.1068	90.0	48	33.94	15.22	0.518	0.4485	1.414
13—14	1.2792	57.6	52	32.14	18.09	0.298	0.5630	1.618
14—15	1.4567	23.0	56	29.04	20.87	0.097	0.7180	1.929
15—16	1.6359	—12.0	60	25.54	23.17	—0.39	0.9070	2.350

will only be identical with $\triangle \beta = Nb.a/EI$ if

$$p = \sqrt{\frac{N}{N^*}\frac{EI^*}{EI}} \left(= \sqrt{\frac{100 \,.\, EI^*}{N^* \,.\, 25000}} = \sqrt{\frac{EI^*}{250\, N^*}} \right) \ldots \quad (3)$$

Consequently fig. 3 holds for every girder and every load P. The results which under the assumption $N = 100$ kg, $EI = 25000$ kg cm^2, can be read from this figure are assembled in the first five columns of table 1. The last three columns contain those dimensionless quantities which are essential for the problem. The quantity λ has been chosen to enable a comparison with the results of SONNTAG. As can be seen from fig. 4, in which λ, f/l and l_e/l are plotted against β, and in which the results obtained by SONNTAG have been marked by points, the agreement is a remarkably good one. It may be remarked, that β has been chosen as the independent variant only to simplify the connection with SONNTAG's work. A more satisfying representation, as far as λ is regarded, is given in fig. 5, where λ is represented as function of the deflection-ratio f/l. (In both figures 4 and 5 the middle curve λ is in discussion here; in the last one SONNTAG's results are represented by the dotted line). It follows from this figure that only to a certain amount of the load P, the girder is in stable equilibrium. The load for which instability occurs can easily be read from the graph.

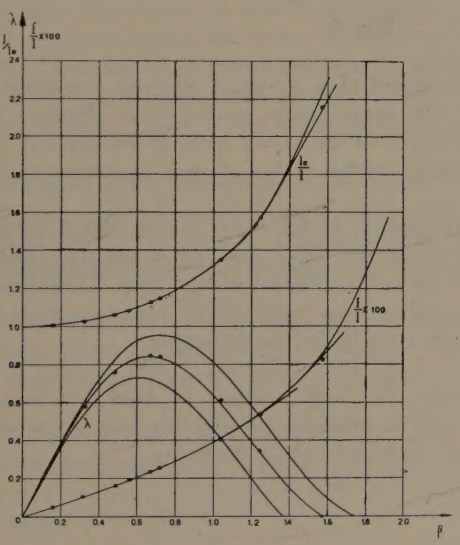

Fig. 4.

3. *Additional remarks.* Some generalizing remarks can be made.

a. The construction given in this paper can be extended without any difficulty to symmetrical girders of varying flexural rigidity.

b. It can as well be extended to cases where friction becomes into being during the gliding motion of the girder over its supports. According to the two possible directions of gliding frictional forces of the magnitude $\pm \mu N$ ($\mu =$ coefficient of friction) have to be introduced. The value of N being assumed, the influence of these frictional forces can be brought into account in the very same manner as that of the reaction N, and the whole construction can therefore be performed as in fig. 3. The required loading force P is given by $P = 2N\ (\cos\beta \pm \mu \sin\beta)$. It has been indicated in the left upper corner of fig. 3, how (for $\mu = 0.2$) P could be found as soon as the central line would be known. If the direction of P would be given by n_8, $P_{7,8}$ would be equal either to

$S_1 - (7,8)$ or to $S_2 - (7,8)$. From the figures 4 and 5, in which for both gliding directions the graphs for λ corresponding to $\mu = 0{,}2$ are represented, it is seen that the frictional influence is considerable, and therefore it does not surprise that SONNTAG's experiments in which friction was not sufficiently eliminated did not very well agree with his theoretical results, which no doubt are very accurate.

c. Whereas SONNTAG's deductions only hold true as far as the central line of the

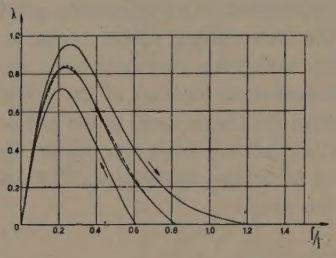

Fig. 5.

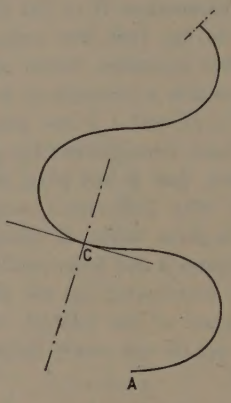

Fig. 6.

girder can be approximated in a sufficient way by a suitable chosen circle-arch, the method proposed in this paper allows an important extension in that the construction of fig. 3 can be continued as far as we wish. If we proceed in the prescribed way a line arises as represented in fig. 6, the obvious properties of symmetry of which can be easily proved.

The normal of any point C of this curve can be looked upon as a line of symmetry of a possible central line of the girder, one half of which is represented by the arch AC. It is worth while to draw the different central lines, corresponding to a great number of points C; for this will give an insight in the numerous ways in which the girder can deform. This, however, must be left to the reader.

4. *Acknowledgment.* The author wishes to thank his former assistant, Mr. A. D. DE PATER to whom he is indebted for valuable help in drawing the figures, and in discussing the matters, mentioned under 3*b* and *c*.

Mathematics — *Die repräsentierende Menge der stetigen Funktionen des Einheitskontinuums.* Von Prof. L. E. J. BROUWER.

(Communicated at the meeting of April 25, 1942.)

Wenn eine unbegrenzte Folge von Operationen F_ν ausgeführt wird, deren jede darin besteht, dass jedem Punktkerne des Einheitskontinuums ein λ-Intervall [1]) zugeordnet wird, und zwar in solcher Weise, dass für jeden Punktkern das von $F_{\nu+1}$ erzeugte Intervall jedesmal im von F_ν erzeugten Intervall im engern Sinne enthalten ist, so soll diese Operationsfolge eine *volle Freifunktion des Einheitskontinuums* heissen. Offenbar ordnet sie jedem Punktkern des Einheitskontinuums einen Punktkern des Linearkontinuums zu. Demgemäss bilden die früher eingeführten *vollen Funktionen des Einheitskontinuums* [2]) einen Spezialfall der vollen Freifunktionen des Einheitskontinuums. Der früher für die vollen Funktionen geführte Beweis der gleichmässigen Stetigkeit [3]) lässt sich aber ungeändert für die vollen Freifunktionen übernehmen.

Wenn für gegebenes m und n jedem $\varkappa(m)$-Intervall [1]) K des Einheitskontinuums ein $\lambda(\nu)$-Intervall [1]) (ν variabel mit dem Minimum n) $\varphi(K)$ so zugeordnet ist, dass aneinander grenzenden K entweder identische oder ein gemeinsames Segment besitzende $\varphi(K)$ entsprechen, so nennen wir diese Zuordnung ein *m n-Treppenpolygon.* Als Spezialfall definieren wir einen *m n-Treppenblock*, indem wir weiter fordern dass jedes $\nu = n$ ist.

Für $m_2 > m_1$ und $n_2 > n_1$ soll das m_2 n_2-Treppenpolygon T_2 im m_1 n_1-Treppenpolygon T_1 *eingelagert* heissen, wenn für ein beliebiges $\varkappa$-Intervall K_1 von T_1 und ein beliebiges $\varkappa$-Intervall K_2 von T_2 die Beziehung $K_2 \subset K_1$ nach sich zieht, dass $\varphi_2(K_2)$ einen echten Teil von $\varphi_1(K_1)$ bildet.

Unter einer *Treppenfunktion* (bzw. *Blockfunktion*) verstehen wir eine unbegrenzte Folge von Treppenpolygonen (bzw. Treppenblöcken), deren jedes in dem ihm vorangehenden eingelagert ist. Offenbar ordnet eine Treppenfunktion (bzw. Blockfunktion) jedem Punktkern des Einheitskontinuums einen Punktkern des Linearkontinuums zu.

Wir nennen eine volle Freifunktion des Einheitskontinuums und eine Treppenfunktion oder Blockfunktion einander *gleich* und sagen, dass sie einander *repräsentieren*, wenn sie jedem Punktkern des Einheitskontinuums denselben Punktkern des Linearkontinuums zuordnen.

Man beweist unschwer, dass *erstens* jede Treppenfunktion bzw. Blockfunktion einer vollen Freifunktion des Einheitskontinuums gleich ist, *zweitens* jede volle Freifunktion des Einheitskontinuums sich durch eine Treppenfunktion und sogar durch eine Blockfunktion repräsentieren lässt.

Nun sind aber die Spezies aller Treppenblöcke (ebenso wie die Spezies aller Treppenpolygone) und die Spezies der Treppenblöcke, welche in einem gegebenen Treppenblock eingelagert werden können (ebenso wie die Spezies der Treppenpolygone, welche in einem gegebenen Treppenpolygon eingelagert werden können) abzählbar unendlich. Nach dem vorhergehenden folgt hieraus unmittelbar, dass *die Spezies der vollen Freifunktionen des Einheitskontinuums sich durch eine non-finite Menge repräsentieren lässt*, nämlich durch die Menge welche entsteht, wenn zunächst ein beliebiger Treppenblock T_1 gewählt wird, und sodann der Reihe nach für jede natürliche Zahl ν ein beliebiger in T_ν eingelagerter Treppenblock $T_{\nu+1}$.

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1) Mathem. Annalen, Bd. **93**, S. 253 (1925); Bd. **97**, S. 60 (1926).
2) Mathem. Annalen, Bd. **97**, S. 62 (1926).
3) Mathem. Annalen, Bd. **97**, S. 66, 67 (1926).

Physiology. — *The significance of the interstitial cells in the development of the motor end-plates in mammals (talpa, mus, homo, lepus).* By J. BOEKE, LL.D., M.D., Utrecht.

(Communicated at the meeting of April 25, 1942.)

In a former communication (Proceedings of the meeting of February 28, 1942) I discussed the problem of the interstitial cells (neurones interstitiels of CAJAL) and demonstrated that they are not only found lying everywhere at the end of the sympathetic plexuses [1]), but that homologous elements may be found also in the endformation of the spino-cerebral nerves.

As it was mentioned allready in a lecture, delivered in England (Universities of London and Oxford) in 1937, a lecture which was published by the Oxford University Press in 1940, it was suggested in the communication mentioned above, that the cells of the core of the sensory corpuscles, which are in a syncytial connexion with the neurofibrillar ending and its surrounding neuroplasma, and even part of the constituents of the motor end-plates of the cross-striated muscle-fibres might be homologized with the interstitial cells of the sympathetic endformation. It was suggested in 1937 that these interstitial elements formed the "neurohumoral region" of these afferent and efferent nerve-endings (as of the sympathetic endformation), in which the humoral energy, necessary to transform the nervous stimulus, is produced.

In the communication in the meeting of February the development of the motor end-plates was sketched and the derivation of the nuclei of the soleplate known as the nuclei of the arborisation (noyaux de l'arborisation de RANVIER) from ingrowing elements of the nerve-fibres was described. Here I may be allowed to describe this development more fully and in more details.

The development of the motor end-plate with special reference to the part played by the different nuclei in the formation of the sole-plate and of the establishment of the connexion of the nerve-endings and the protoplasm of the muscle-fibre during this development was studied especially in the tongue of embryos of the mole and of human embryos of 4½ and 5½ months, the developing motor end-plates of young mice and of rabbit embryos, which showed the same details, being only used to confirm the conclusions drawn from the study of embryos of the mole and human embryos.

The embryos studied were fixated in a mixture of formalin and alcohol or in neutralized formalin (10%). They were impregnated after the method of BIELSCHOWSKY and afterwards treated with chloride of gold. In this way we get not only a splendid black

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[1]) Most modern writers, as far back as LA VILLA in 1898, BETHE in 1903, MUENCH and SCHOCK 1910, LEONTOWITCH and ERIK MÜLLER in 1921, have accepted the view of CAJAL, that they are of nervous origin and nature, and in 1926 and 1928 LAWRENTJEW and VAN ESVELD showed that they are the normal endings of the plexus-elements. This view was accepted for instance by SCHABADASCH in 1934, BOEKE 1933, 1935, LEEUWE 1937, MEYLING 1938, TINEL 1938, PEY–LIN LI, 1940, a.o.. Even STOEHR has been compelled in 1941 to make room for the interstitial cells in his conception of the "terminal reticulum" (which in its original conception was entirely devoid of nuclei and cellular structures), be it together with the other small cells of the sympathetic ganglia, the "Nebenzellen" of KOHN etc.; but in my opinion these small cells are of an entirely different nature and have nothing to do with the interstitial elements lying at the end of the sympathetic plexus (see the paper in the Acta Morphologiae Neerlandica).

impregnation of the neurofibrillar structures but also an excellent colouring of the nuclei and the protoplasma of the nervous elements and of the surrounding elements of the different tissues. This enables us to study not only the development of the neurofibrillar structure of the motor nerves and their endings but also the structure of the muscle-fibres and the development of the sole-plate, the different nuclei and their movements, and the form and extension of the protoplasma of the interstitial elements during the formation of the motor endings. The importance of this for the study of these elements needs no demonstration. Without it a study of these elements is impossible. In the excellent and exhaustive description of the development of the motor endings by TELLO (1917), in which he studied and pictured preparations made by the method of CAJAL, he only describes the development of the motor nerves and the neurofibrillar structures of the outgrowing nerve-fibres, because according to his own statement "the protoplasma was not stained in his preparations and therefore could not be studied." (l.c. page 172). However excellent therefore the observations recorded in his paper might be, they refer unevitably only to one small part of the question. For when we study closely the beautiful drawings which are reproduced in his paper, we get the impression that TELLO has seen exactly what we are going to describe here, only he did not pay any attention to it because he could not trace the different elements without the protoplasm being stained and visible, and because as a true neuronist he was convinced that the nerve-fibres develop and grow as free independent neurofibrillar structures without any connexion with the surrounding neighbouring elements; for him therefore these surrounding elements were not of any importance, and he simply mentions the position of the nuclei at the end of the nerve-terminations at the time that these are seen to push their way into the sarcoplasma of the muscle-fibres (that is at the time the "miofibras" develop by longitudinal fissure of the primary "miotubos", and before a membranous sarcolemma was formed, by which the growing muscle-fibre gets its individuality and its independency), without paying any further attention to them.

According to his description the muscle-fibres develop from the primary muscle-tubes (miotubos) by a longitudinal fissure, the nerve-fibres grow into the connective tissue of the muscular masses, they divide and form at their end the bulbous swellings known from the regeneration-process of the nerve-fibres and so characteristic for the growing fibres of the embryonic nerves (c.f. SPEIDEL, 1936). These terminal swellings (cônes de croissance, which after the description of CAJAL "act like battering-rams against the surrounding elements") lie free in the connective tissue between the developing muscle-fibres; before a sarcolemma is formed (c.f. BARDEEN, 1907) they grow into the developing muscle-fibres there where a multiplication of the nuclei of the muscle-fibre indicates the place of the future sole-plate of the fibre. According to TELLO the growing nerve-fibres are attracted by these sole-plate-formations (by neurotropism), enter them and form in this sarcoplasma of the sole-plate their terminal ramifications (by neurocladism).

According to this description the developing nerve-fibres act entirely by their own force. They may be followed in their course by the elements of the nerve-sheath, the lemmoblasts, which envelop them later on, but they sprout and push their way independently of the surrounding elements. According to TELLO there is no trace of any conducting elements. Here he follows the line of most authors.

It is to be regretted that the beautiful drawings by which the paper of TELLO is illustrated, are for the greater part sketched under a low power and are meant only as pictures illustrating the general mode of development. Only in some cases they give minor details, and then we see (for instance in fig. 15 and 33) that there where the terminal bulbs of the ingrowing nerve-fibres touch the muscle-fibres a curious-shaped nucleus is found lying close to the nerve-ending, which nucleus according to TELLO belongs to the muscle-fibre itself and simply indicates the beginning of the process of formation of the nuclei of the sole-plate and of their ultimate arrangement (TELLO, page 172).

As I mentioned allready, TELLO is not the only author, who mentions the presence

of nuclei in the region of the muscle-fibre where the motor end-plate is going to be formed. According to nearly all the writers on the development of the motor nerve-endings on striated muscle-fibres (for instance KÜHNE, KRAUSE, TRINCHESE, LONDON & PESKER, MAYS a.o.) the developing motor nerves come into contact with the muscle-fibres at a spot, where one or more nuclei are found lying under the sarcolemma ("nuclear region", "Kerngebiet", "besonders dichte Anhäufung von Kernen" etc.). They are convinced that these nuclei belong to the muscle-fibre and not to the ingrowing nerve-fibres. Some of them however (f.i. TRINCHESE) go more into details and describe the nuclei present as belonging to the fundamental nuclei of the sole-plate (Grundkern, TRINCHESE) while according to others (f.i. MAYS) the nucleus lying at the spot, where the nerve-ending is going to develop, is a nucleus of the arborisation ("Geästkern"), which disappears however after the motor end-plate is fully developed. But after all they are convinced that the ingrowing nerves are devoid of nuclei, and the nuclei they describe belong to the muscle-fibres; TELLO however pictures in his figures of the ingrowing nerves distinct elongated nuclei between the bundles of nerve-fibres, without paying any attention to them.

As TELLO is the author, who describes the development of the motor nerve-endings with the greatest accuracy, I may be allowed to restrict myself to discuss here only his description and figures.

As it was mentioned in my former communication (Febr. 1942, page 213) this assertion that the nuclei visible around the ingrowing nerve-endings belong undeniably to the muscle-fibre, is liable to be critisized, and it is still an open question, whether the ingrowing nerve-fibres are in reality devoid of accompanying cellular elements, as it was described so convincingly by HARRISON (R. G. HARRISON, Neuroblast versus sheath-cell, Journ. of comp. Neurology, Vol. **37**, 1923). Where the sympathetic nerve elements in reality come from, is still unknown. As it was demonstrated by LEEUWE (see my XII. Innervationsstudie, Acta Morph. Neerl. 1942), the interstitial cells are derived from the ganglia of the sympathetic chain and plexus and grow out from them as distinct cellular elements, which remain in a syncytial arrangement and in connexion with the true ganglion cells. They swerve out into the surrounding tissues and accompany the outgrowing nerve-fibres. Where even their existence has been denied and in embryonic tissues most writers could not find them, allthough they must have been there, it is premature to deny the presence of cellular elements accompanying the embryonic nerve-fibres on their way to their ultimate destination, simply because we could not find them. In my opinion the nuclei (one or two in number) figured by TELLO as lying at the top of the ingrowing nerves, do *not* belong to the muscle-fibre but to the ingrowing nerve-fibres themselves, as they clearly surpass the outlines of the growing muscle-fibres. In my BIELSCHOWSKY-preparations (talpa and human embryos of 4½ and 5½ months and new-born mice) not only these nuclei but also the surrounding protoplasma was stained and therefore clearly visible; the nuclei with their surrounding protoplasma belong undoubtedly to the ingrowing nerve-fibres. They surround the nervous terminal bulbs and are lying outside the muscle-fibres as distinct conducting elements, constituting, as far as could be studied in the preparations, the syncytial terminal elements of the ingrowing nerve-fibres. In longitudinal and cross-sections of developing muscle-fibres I could study these details with the greatest accuracy.

These nuclei are allways lying at the end of the growing nerve-fibres, and when these nerve-fibres reach their destination, the muscle-fibres, the nuclei are allways lying at the side of the terminal bulbs or terminal rings of the nerve-fibre which is turned away from the muscle-fibre and never between the bulbs and the sarcoplasma of the muscle-fibre itself; they cover the terminal arborisations which are developing now. These terminal ramifications of the growing motor nerve-endings are formed inside the protoplasma of these terminal conductive elements. The conducting elements flatten themselves against the surface of the muscle-fibre, their protoplasma fuses with the sarcoplasma of the muscle-fibre, becomes a part of the sarcoplasmic sole-plate, and

only then a distinct sarcolemma is formed, which envelops both the flattened conductive element and the sarcoplasma of the muscle-fibre with which it was fused, and in this way the definite sole-plate and the motor nerve-ending is formed.

In the sections of talpa-embryos of different sizes (18—31 m.M.) and of the developing muscle-fibres of human embryos of 4½ and 5½ months I could follow the different phases of this process of fusion with great clearness. In cross-sections through the muscle-fibres and in longitudinal sections there where the terminal ramifications of the nerve-fibres and their conducting elements were lying "en profil" at the side of the muscle-fibre, this process of fusion and of the formation of the terminal ramifications of the nerve-ending inside the protoplasma of these conducting elements was to be followed with the utmost clearness; after the fusion of the two elements and the formation of the definite sole-plate with its accumulation of nuclei in the sarcoplasma of the muscle-fibre the terminal ramifications of the nerve-ending seem to extend throughout the whole mass of protoplasma (sarcoplasma); at least I could not find a separation or demarcation-line between the sarcoplasma of the original muscle-fibre and the protoplasma of the conducting element fused with it. But here I could call attention to the curious fact, described allready by KÜHNE and by the other histologists of the former century and affirmed by nearly every writer on the subject, that the terminal arborescence of the motor end-plate *never* comes into contact with the myofibrillar structure itself, but that in every case it remains separated from the contractile myofibrillar structure itself by a thin but distinct layer of sarcoplasma, in which the periterminal network (the receptive substance of LANGLEY) furnishes an intermediate substance. Perhaps this curious fact finds a solution in the observations just recorded.

That the protoplasm of two different elements fuses to form a living and functional unity, is not rare or uncommon. We see it everywhere in the organism. The free cells of the embryonic mesenchym fuse to form a syncytium; the cells of the developing heart-muscle fuse in the same way (GODLEWSKI). SPEIDEL for instance (1932) describes how during the development of the nerves in the growing *living* organism "sheath cells may transfer readily from one nerve to another after temporary or permanent anastomoses. They may also bridge the gap and effect transfer between two nerves merely placed in close proximity without any anastomosis," (SPEIDEL, 1932, page 306). The interstitial elements of he sympathetic endformation are everywhere in a syncytial connexion, the elements of the core of the sensory corpuscles exhibit the same syncytial arrangement, and in their protoplasm the conducting periterminal network is formed. Every one who has seen the splendid films of growing tissue-cultures by CANTI, remembers how in these cultures of fibroblasts forming a syncytial reticulum distinct cells which for a long time creep about as distinct independent elements, all at once wriggle themselves into the reticulum formed by the fibroblasts and become part of it. We could multiply these comparable cases, but I may leave it at these few examples.

Whenever a terminal nervous arborisation on the muscle-fibre is formed, these nuclei are allways present, and they may be readily distinguished from the pale elongated nuclei of the muscle-fibre itself by their form, their size and their structure [1]). In older stages of the embryonic development, when the accumulation of the nuclei of the sole-plate begins to show itself and a distinct sarcolemma is formed, the difference between the nuclei of the arborisation and the fundamental nuclei belonging to the muscle-fibre itself often becomes still more prominent, and even in young mice (one to three days old) the two sorts of nuclei are readily to be distinguished, as it was pictured and described in my former papers. These details however will be described in a separate and extensive paper; here only the outlines of the problem investigated can be recorded.

The elements just described belong to the nerve-fibres and are of nervous origin;

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[1]) In a paper appearing in the Acta Morphologiae Neerlandica these details will be described more fully and with ample illustrations. Here of course only the outlines of my investigations can be recorded.

apparently they swerve out at the end of the outgrowing nerve-fibres until they reach the developing muscle-fibres. This reminds us of the observation recorded in my former paper (XII. Innervationsstudy in the Acta Morphologiae Neerlandica), that according to the investigation of LEEUWE the interstitial elements swerve out from the growing accumulations of the ganglion cells of the sympathetic plexus and become connected with them later on by their neurofibrillar structure. It seems to me that we are entirely justified to regard the elements described as the homologa of the interstitial syncytial elements of the sympathetic endformation.

But after all we need to be very careful with regard to their origin. As it was discussed briefly on page 5 of this communication we still know very little about the cells accompanying the outgrowing nerve-fibres in the living organism, and the question whether the interstitial cells and the other small elements of the ganglia (especially of the sympathetic system) are all of them of nervous origin is still under discussion; we only need to point to the critique of HERZOG and GÜNTHER of the work on these elements of STÖHR (1941). To follow the elements just described through all the phases of their development and wanderings will have to be done still. And the way of these investigations is full of pitfalls. For instance in several sections of the growing musculature, especially in the human embryos which I could study, I often encountered between the developing muscle-fibres rather large cells with branching processes and a granular protoplasm full of small elongated and oval granules, which are stained black or dark brown with silver (BIELSCHOWSKY-preparations). These cells were not only seen lying close to the muscle-fibres, but they seem to fuse with the sarcolemma by means of their cell-processes. They are seemingly of mesodermal origin and belong to the different elements of the connective tissue surrounding the growing muscle-fibres; they look like cells capable of ameboid motion, have large and often long processes and a granular protoplasma, and resemble most the resting wandering cells of MAXIMOW, the histiocytes or macrophages of the connective tissue, but in a peculiar form. I mention them here, because at first sight one might regard them perhaps as interstitial cells because of their form and the granular structure of their protoplasm. Closer study however reveals them as connective tissue elements, never in connexion with nervous elements, but often in connexion with the wall of the growing blood capillaries; in the more elaborate paper I hope to describe them more fully and to show the details of their protoplasmic structure in accurate figures. In the same paper the origin of the interstitial elements described here will be discussed more fully, together with the question, whether they belong to the sheath cells or to the neuroblasts, a question which was discussed allready in my former paper, in a discussion which led to the conclusion, that a sharp line of demarcation between neuroblasts and sheath cells is not to be drawn and that the interstitial cells form a distinct intermediate structure between them which has a sharply defined chemical function (the neuro-humoral region). There they will be compared with the interstitial elements in the connective tissue of the iris and of the cornea, and in this way their full value in the organism will be reviewed.

Geophysics. — *On Surface Waves in a Stratisfied Medium.* II. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS.)

(Communicated at the meeting of March 28, 1942.)

§ 4. *The waves in an isolated layer.*

The wave system possible in an isolated layer consists of the waves existing in a superficial layer (fig. 2). As the tension at the free surfaces is equal to zero we have

$$\text{at } z=d \begin{cases} n e^{-i\alpha}\cos 2r \,.\, A_e - e^{-i\beta}\sin 2r \,.\, \mathfrak{A}_e + n e^{i\alpha}\cos 2r \,.\, A_r - e^{i\beta}\sin 2r \,.\, \mathfrak{A}_r = 0 \\ e^{-i\alpha}\sin 2i \,.\, A_e + n e^{-i\beta}\cos 2r \,.\, \mathfrak{A}_e - e^{i\alpha}\sin 2i \,.\, A_e - n e^{i\beta}\cos 2r \,.\, \mathfrak{A}_r = 0 \end{cases}$$

$$\text{at } z=0 \begin{cases} n\cos 2r \,.\, A_e - \sin 2r \,.\, \mathfrak{A}_r + n\cos 2r \,.\, A_r - \sin 2r \,.\, \mathfrak{A}_r = 0 \\ \sin 2i \,.\, A_e + n\cos 2r \,.\, \mathfrak{A}_r - \sin 2i \,.\, A_e - n\cos 2r \,.\, \mathfrak{A}_r = 0 \end{cases}$$

where $\alpha = h\,d\cos i$ and $\beta = k\,d\cos r$.

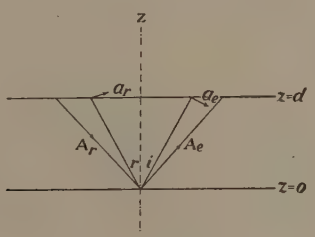

Fig. 3

The wave system $\{A_e\ A_r\ \mathfrak{A}_e\ \mathfrak{A}_r\}$ is therefore only then possible if

$$\begin{vmatrix} n\cos 2r \,.\, e^{-i\alpha} & -\sin 2r \,.\, e^{-i\beta} & +n\cos 2r \,.\, e^{i\alpha} & -\sin 2r \,.\, e^{i\beta} \\ \sin 2i \,.\, e^{-i\alpha} & +n\cos 2r \,.\, e^{-i\beta} & -\sin 2i \,.\, e^{i\alpha} & -n\cos 2r \,.\, e^{i\beta} \\ n\cos 2r & -\sin 2r & n\cos 2r & -\sin 2r \\ \sin 2i & +n\cos 2r & -\sin 2i & -n\cos 2r \end{vmatrix} = 0,$$

or

$$(\sin^2 2i \sin^2 2r + n^4 \cos^4 2r)\sin\alpha\sin\beta + 2\sin 2i \sin 2r \,.\, n^2\cos^2 2r(1-\cos\alpha\cos\beta) = 0.$$

or

$$\left(\sin 2i \sin 2r + \frac{tg\,{}^1\!/_2\,\alpha}{tg\,{}^1\!/_2\,\beta} \,.\, n^2\cos^2 2r\right)\left(\sin 2i \sin 2r + \frac{tg\,{}^1\!/_2\,\beta}{tg\,{}^1\!/_2\,\alpha} \,.\, n^2\cos^2 2r\right) = 0.$$

Hence the equation of the elastic vibrations possible in an isolated layer is

$$\frac{\sin 2i \sin 2r}{n^2\cos^2 2r} = -\frac{tg\,{}^1\!/_2\,\alpha}{tg\,{}^1\!/_2\,\beta} \quad \text{and} \quad \frac{\sin 2i \sin 2r}{n^2\cos^2 2r} = -\frac{tg\,{}^1\!/_2\,\beta}{tg\,{}^1\!/_2\,\alpha}.$$

Putting $\sin r = \frac{1}{\sqrt{\zeta}}$ we get

$$\frac{4\sqrt{(\nu\zeta-1)(\zeta-1)}}{(\zeta-2)^2} = -\left(\frac{tg\,{}^1/_2\,\alpha}{tg\,{}^1/_2\,\beta}\right)^{\pm 1}$$

where

$$\alpha = kd\sqrt{\frac{\nu\zeta-1}{\zeta}} \text{ and } \beta = kd\sqrt{\frac{\zeta-1}{\zeta}}. \quad . \quad . \quad . \quad . \quad (2^a)$$

If $\zeta > 1/\nu$ all waves are periodical. If $1 < \zeta < 1/\nu$ the longitudinal waves are damped and the transverse waves periodical; the equation becomes

$$\frac{4\sqrt{(1-\nu\zeta)(\zeta-1)}}{(\zeta-2)^2} = -\frac{tgh\,{}^1/_2\,kd\sqrt{\frac{1-\nu\zeta}{\zeta}}}{tg\,{}^1/_2\,kd\sqrt{\frac{\zeta-1}{\zeta}}} \text{ and } \frac{4\sqrt{(1-\nu\zeta)(\zeta-1)}}{(\zeta-2)^2} = +\frac{tg\,{}^1/_2\,kd\sqrt{\frac{\zeta-1}{\zeta}}}{tgh\,{}^1/_2\,kd\sqrt{\frac{1-\nu\zeta}{\zeta}}} \quad (2^b)$$

In the case $\zeta < 1$ all waves are damped; then the equation is

$$\frac{4\sqrt{(1-\nu\zeta)(1-\zeta)}}{(2-\zeta)^2} = +\left\{\frac{tgh\,{}^1/_2\,kd\sqrt{\frac{1-\nu\zeta}{\zeta}}}{tgh\,{}^1/_2\,kd\sqrt{\frac{1-\zeta}{\zeta}}}\right\}^{\pm 1}. \quad . \quad . \quad . \quad (2^c)$$

In figure 4 we have plotted the values of the roots of equations (b) and (c) against $N = kd$; as $k = \frac{\mathfrak{V}}{p} = \frac{2\pi}{L}$, L being the transversal wave length, $N = 2\pi\frac{d}{L}$. These roots are also to be found in table I.

This result was already obtained by LAMB, who discussed in a paper published in 1916 the damped and semi-periodical wave systems in an elastic plate [11]). The solutions given in table I are calculated from the values computated by LAMB. Th solutions corresponding

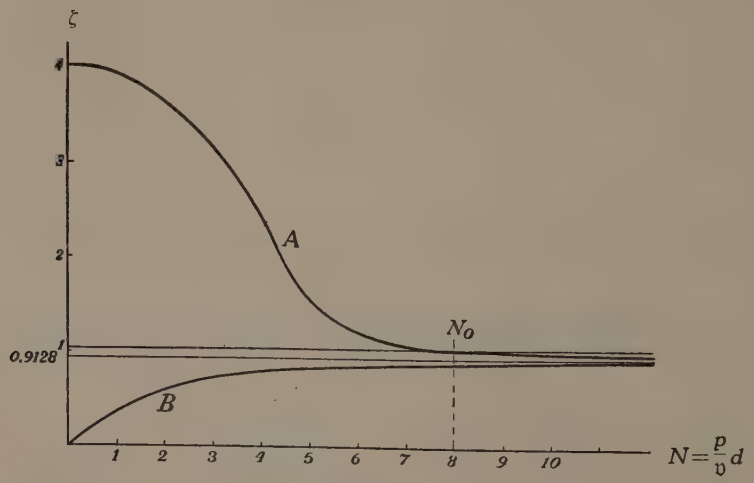

Fig. 4

to the periodical wave systems are not relevant to the present problem. As the vibrations of an elastic plate will be discussed in a subsequent paper, we will not further pursue the investigation of the properties of these waves, having already ascertained the data needed for the solution of our main problem.

TABLE I.

Curve A				Curve B			
ζ	N	ζ	N	ζ	N	ζ	N
0.9128	∞	2	4.443	0.9128	∞	0.437	1.111
0.958	9.292	2.440	3.964	0.878	7.871	0.360	0.847
1	8.004	2.961	3.327	0.797	4.151	0.278	0.593
1.010	7.786	3.889	1.147	0.640	2.203	0.190	0.379
1.160	6.208	3.956	0.694	0.578	1.776	0.097	0.187
1.360	5.435	3.992	0.371	0.510	1.421	0	0
1.641	4.900	4	0	—	—	—	—

Par. 5. The existence of the generalised RAYLEIGH *and* STONELEY *waves.*

In this paragraph we shall determine for which values of the material constants equation (1)

$$\left.\begin{aligned}
&(P-Q_2)\,.\,\{4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}\,.\,\varphi-(2-\omega\zeta)^2\,.\,tgh\,\alpha\,tgh\,\beta\}+\\
&+(S-Q_1)\,.\,\{4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}\,.\,tgh\,\alpha\,tgh\,\beta-(2-\omega\zeta)^2\,.\,\varphi\}=\\
&=R_1\,.\,\{4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}\,.\,tgh\,\alpha-(2-\omega\zeta)^2\,.\,tgh\,\beta\}+\\
&+R_2\,.\,\{4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}\,.\,tgh\,\beta-(2-\omega\zeta)^2\,.\,tgh\,\alpha\}+\\
&+\frac{w^2\zeta^2\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}}{cosh\,\alpha\,.\,cosh\,\beta}\{4\sqrt{(1-\zeta)(1-\nu_1\zeta)}-(2-\zeta)^2\}
\end{aligned}\right\}\quad(1)$$

has a real root $\zeta<1$. As we presume all waves to be damped, the irrational terms must be real, so that $\zeta<1$, if $\omega<1$ and $\zeta<1/\omega$, if $\omega>1$.

If ζ is very small then α and β are very large, owing to the factors $\sqrt{\frac{1-\zeta}{\zeta}}$ and $\sqrt{\frac{1-\omega\zeta}{\zeta}}$; hence the functions φ, $tgh\,\alpha$ and $tgh\,\beta$ are nearly equal to 1, while cosh α and cosh β are very large. Equation (1), therefore, changes into the STONELEY equation: $P-Q_2+S-Q_1=R_1+R_2$. In our paper on STONELEY waves [5]) we have demonstrated that the left-hand side of this last equation is always smaller than the right-hand side for small values of ζ.

Consequently equation (1) has a root, if the left-hand side is greater than the right-hand side for $\zeta=1$, if $\omega<1$ and for $\zeta=1/\omega$, if $\omega>1$.

A. $\omega<1$ $(\mathfrak{B}_1<\mathfrak{B}_2)$.

Supposing firstly $\omega<1$, we obtain the just mentioned inequality by substituting $\zeta=1$ in the terms of equation (1); as $S=Q_2=R_1=0$ for $\zeta=1$ we get

$$\left(1-2\frac{\mu_2}{\mu_1}+\omega\frac{\mu_2}{\mu_1}\right)^2 . \{4\sqrt{(1-\omega)(1-\gamma)}\,.\,\varphi_1-(2-\omega)^2\,.\,tgh\,\alpha_1\,tgh\,\beta_1\}-$$

$$-\left(1-2\frac{\mu_2}{\mu_1}\right)^2 .\sqrt{(1-\omega)(1-\gamma)}\,.\{4\sqrt{(1-\omega)(1-\gamma)}\,.\,tgh\,\alpha_1\,tgh\,\beta_1-(2-\omega)^2\varphi_1\}>$$

$$\omega\frac{\mu_2}{\mu_1}\,.\sqrt{(1-\nu_1)(1-\omega)}\,.\{4\sqrt{(1-\omega)(1-\gamma)}\,.\,tgh\,\beta_1-(2-\omega)^2\,.\,tgh\,\alpha_1\}-$$

$$-\frac{\omega^2}{(1-\mu_2/\mu_1)^2}\cdot\frac{\sqrt{(1-\omega)(1-\gamma)}}{\cosh\alpha_1\,.\,\cosh\beta_1}$$

where $\alpha_1=\frac{pd}{\mathfrak{B}_1}\sqrt{1-\gamma}$, $\beta_1=\frac{pd}{\mathfrak{B}_1}\sqrt{1-\omega}$, $\varphi_1=1-\frac{1}{\cosh\alpha_1\cosh\beta_1}$.

Or:

$$\left.\begin{aligned}&\left(\frac{\mu_2}{\mu_1}\right)^2.[\{16(1-\gamma)(1-\omega)+(2-\omega)^4\}tgh\,\alpha_1\,tgh\,\beta_1-8(2-\omega)^2\sqrt{(1-\gamma)(1-\omega)}\,.\,\varphi_1]-\\&-\left(\frac{\mu_2}{\mu_1}\right).[\{16(1-\gamma)(1-\omega)+2(2-\omega)^3\}tgh\,\alpha_1\,tgh\,\beta_1-8(2-\omega)(2-{}^1/_2\omega)\sqrt{(1-\gamma)(1-\omega)}\,.\,\varphi_1-\\&\qquad-4\omega\sqrt{(1-\nu_1)(1-\gamma)}\,tgh\,\beta_1+\omega(2-\omega)^2\sqrt{(1-\nu_1)(1-\omega)}\,tgh\,\alpha_1]\\&+\left[\{4(1-\gamma)(1-\omega)+(2-\omega)^2\}\,tgh\,\alpha_1\,tgh\,\beta_1-\right.\\&\qquad\left.-(8-4\omega+\omega^2)\sqrt{(1-\gamma)(1-\omega)}\,.\,\varphi_1-\frac{\omega^2\sqrt{(1-\gamma)(1-\omega)}}{\cosh\alpha_1\cosh\beta_1}\right]>0\end{aligned}\right\}\quad(3)$$

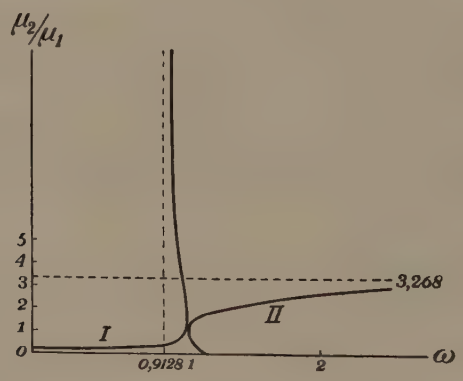

Fig. 5

TABLE II.

Curve *I*					Curve *II*				
ω	μ_2/μ_1		ω	μ_2/μ_1	ω	μ_2/μ_1		ω	μ_2/μ_1
1	1		0.840	0.506	1	1		1.190	1.976
0.999	1.382 ;	0.805	0.750	0.464	1.001	0.724 ;	1.242	1.333	2.155
0.990	2.072 ;	0.700	0.640	0.434	1.010	0.483 ;	1.429	1.562	2.304
0.960	4.637 ;	0.602	0.200	0.336	1.042	0.216 ;	1.661	5	2.976
0.9128	∞	0.5576	0.100	0.322	1.0955	0	1.7933	10	3.106
0.910	—	0.542	0	0.306	1.099	—	1.845	∞	3.268

Note. It can be shown that the common tangent of the two curves should be parallel to the μ_2/μ_1 axis, as is to be seen in the above diagram. This figure is to replace the corresponding fig. 4 on page 163 of our former paper in Proc. Roy. Ac. Amsterdam **45**, 1942.

In the part of the $\{\omega, \mu_2/\mu_1, N\}$-space, where this inequality holds, equation (1) has one root; in the remaining part this equation has no roots (or two roots). The surface of separation between these regions of the $\{\omega, \mu_2/\mu_1, N\}$-space is represented by the equation $f(\mu_2/\mu_1) = 0$ ... (4), $f(\mu_2/\mu_1)$ being the left-hand side of inequality (3).

We will derive the properties of this surface by studying the shape of the curve of intersection of this surface with the planes $\omega =$ a constant. As these curves are largely determined by their asymptotes, it is necessary to investigate the asymptotes existing at a given value of ω. We have, therefore, to discuss the intersection of the surface (4) with the planes $\mu_2/\mu_1 = \sim$ and $N = \sim$, which is rather simple.

If we take $\mu_2/\mu_1 = \sim$, equation (4) becomes

$$\{16(1-\gamma)(1-\omega) + (2-\omega)^4\} tgh\alpha\, tgh\beta - 8(2-\omega)^2 \sqrt{(1-\gamma)(1-\omega)} = 0$$

or

$$\frac{4\sqrt{(1-\gamma)(1-\omega)}}{(2-\omega)^2} = \left(\frac{tgh\,{}^1/_2\alpha}{tgh\,{}^1/_2\beta}\right)^{\pm 1}$$

which equation has the same form as the period equation (2c) of an isolated layer, if we change ω into ζ.

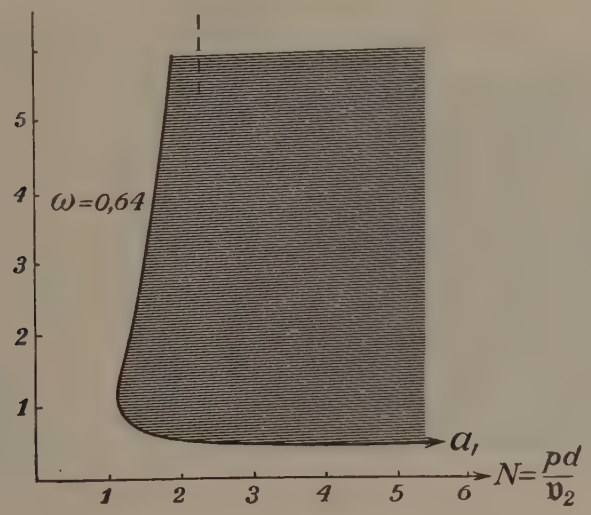

Fig. 6

TABLE III.

N	μ_2/μ_1	N	μ_2/μ_1
1.096	1.162; 1.160	2.32	−19.53 ; 0.482
1.20	1.456; 0.839	2.40	−11.66 ; 0.477
1.60	3.422; 0.589	3.20	− 2.80 ; 0.443
2.20	200.0; 0.500	4.00	− 1.711; 0.427

Putting $N = \sim$ and omitting the factor $\{(2-\omega)^2 - 4\sqrt{(1-\gamma)(1-\omega)}\}$ equation (4) changes into:

$$\left(\frac{\mu_2}{\mu_1}\right)^2 \{(2-\omega)^2 - 4\sqrt{(1-\gamma)(1-\omega)}\} -$$

$$- \left(\frac{\mu_2}{\mu_1}\right) \{2(2-\omega) - 4\sqrt{(1-\gamma)(1-\omega)} + \omega\sqrt{(1-\nu_1)(1-\omega)}\} + \{1 - \sqrt{(1-\gamma)(1-\omega)}\} = 0.$$

Now this equation was studied in our previous paper; figure (5) is the graphic representation of this equation — in the regions between the curves STONELEY waves are possible.

The curves of figure (4) and figure (5) determine for every value of ω the asymptotes of the curves of intersection mentioned above (this curve we shall call henceforward "the boundary curve $\zeta = 1$").

As will be seen from fig. (4) the asymptote of the boundary curve $\zeta = 1$ parallel to the μ_2/μ_1 axis moves with increasing value of ω to greater values of N, and disappears in $N = \sim$, if $\omega = \left(\frac{S_2}{\mathfrak{V}_2}\right)$, that is if $\mathfrak{V}_1 = S_2$, S_2 being the velocity of RAYLEIGH waves in the superficial layer. If $\omega > \left(\frac{S_2}{\mathfrak{V}_2}\right)^2$, or $\omega_1 > S$, the asymptote returns, with growing value of ω, to smaller values of N and reaches the value N_0 if $\omega = 1$ (fig. 6—8). The curve has also an asymptote parallel to the N axis: $\mu_2/\mu_1 = a$; a is determined by fig. (5)

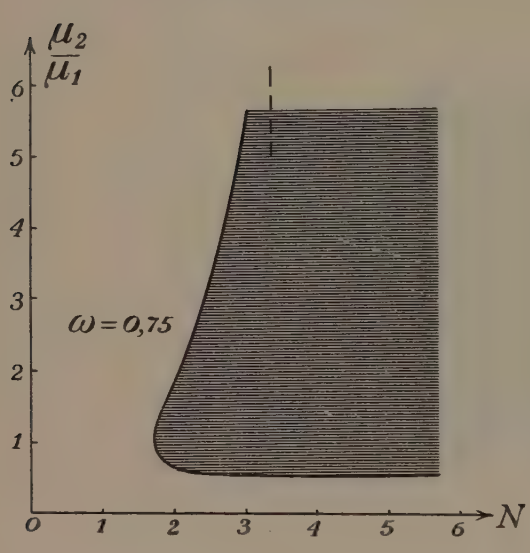

Fig. 7

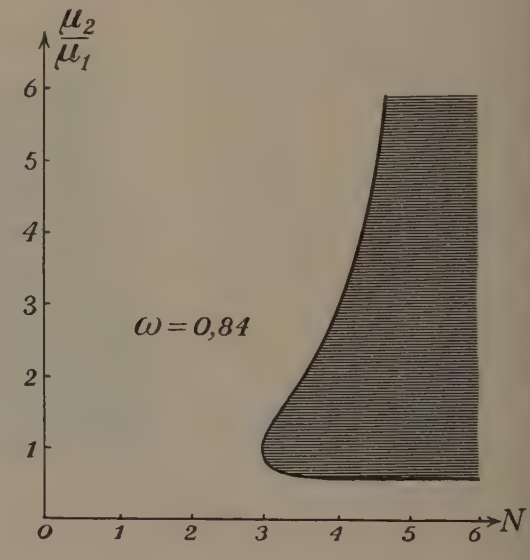

Fig. 8

TABLE IV.

N	μ_2/μ_1		N	μ_2/μ_1	
1.732	1.122	0.946	3.031	9.23	0.523
2.165	2.103	0.634	3.291	39.7	0.500
2.382	2.688	0.593	3.464	—36.2	0.499
2.598	3.639	0.561	4.330	—3.86	0.498

TABLE V.

N	μ_2/μ_1	
3.030	0.970	0.967
3.208	1.202	0.633
3.666	2.022	0.625
4.582	4.86	0.557
5.041	18.5	0.556

for every value of ω. This asymptote moves with increasing value of ω to greater values of μ_2/μ_1; if $\omega = \left(\frac{S_2}{\mathfrak{B}_2}\right)^2$ a second asymptote $\mu_2/\mu_1 = a_2$ appears, which moves in opposite direction to the first asymptote. The two asymptotes coincide in $\mu_2/\mu_1 = 1$ if $\omega = 1$ (fig. 9—10).

With some numerical computations (the results of which are given in tables III—VII) the shape of the curves can be readily found.

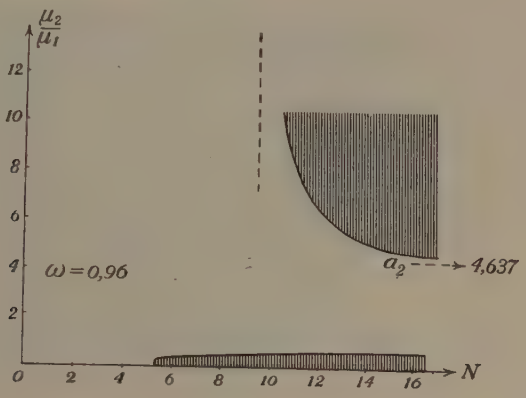

Fig. 9

TABLE VI.

N	μ_2/μ_1	N	μ_2/μ_1
5.38	0.303; 0.300	9.31	2106.; 0.580
5.88	— 0.216; 0.400	9.41	82.8 ; 0.583
6.86	— 1.412; 0.498	9.60	34.1 ; 0.580
7.84	— 3.853; 0.533	11.76	6.80; 0.598
8.82	—16.30 ; 0.574	12.74	5.88; 0.601
9.30	— ∼ ; 0.580	∼	4.64; 0.601

In order to determine the number of roots of equation (1) in the two parts of the $\{\omega, \mu_2/\mu_1, N\}$-space, separated by the surface (4), we pursue the following line of reasoning:

It will be obvious that this problem is solved if we know the number of roots at some special points of this functional space. It is convenient to choose the points of the plane $N = \sim$. If $N = \sim$ equation (1) breaks up into the STONELEY equation and the equation:

$$4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)} - (2-\omega\zeta)^2 = 0.$$

Putting $\omega\zeta = \eta$ this equation becomes the RAYLEIGH equation for a semi-infinite body composed of the same material as the superficial layer; this equation has a root $\eta = \frac{S_2}{\mathfrak{B}_2}$. Hence the "RAYLEIGH-root" ζ_R of equation (1) is equal to $\frac{S_2}{\mathfrak{B}_2} \times \frac{1}{\omega}$ or $\frac{S_2}{\mathfrak{B}_1}$. This, however, is a root of equation (1) only if $\zeta_R < 1$, or $S_2 < \mathfrak{B}_1$; that is to say if the velocity of the transversal waves in the substratum is larger than the RAYLEIGH velocity in the layer.

As we have discussed already [5]) for which values of μ_2/μ_1 a STONELEY-root (ζ_S) is possible (see also fig. 5), we can ascertain for every value of the material constants how many roots equation (1) yields if $N = \sim$.

We proceed now with the application of this method to all possible cases:

a) $\mathfrak{V}_1 < S_2 \, (\omega < S_2/\mathfrak{V}_2)$.

If $\mathfrak{V}_1 < S_2$ the RAYLEIGH-root does not satisfy equation (1); a STONELEY-root is only possible if $\mu_2/\mu_1 < a_1$. Hence at $N = \sim$ equation (1) has one root if $\mu_2/\mu_1 < a_1$ and no roots if $\mu_2/\mu_1 > a_1$.

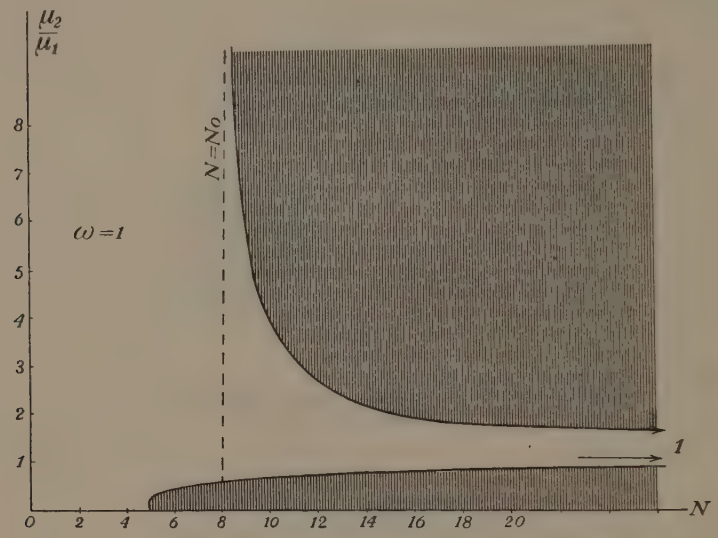

Fig. 10

TABLE VII.

N	μ_2/μ_1	N	μ_2/μ_1	N	μ_2/μ_1
4.83	0.156; 0.223	8	$\sim$; 0.600	16	1.902; 0.723
4.84	0.135; 0.237	9	6.372; 0.628	18	1.762; 0.738
5	—0.021; 0.308	10	3.851; 0.649	20	1.667; 0.750
6	—1.001; 0.498	12	2.569; 0.681	$\sim$	1 ; 1
7	—3.561; 0.562	14	2.129; 0.705		

Consequently the part of the $\{\mu_2/\mu_1, N\}$-plane where one root is possible is the region, which at large values of N lies under the asymptote $N = a_1$. In the remaining part (the horizontally shaded area in fig. 6—8) no root is possible.

With increasing values of ω the first region expands more and more, covering at last the whole of the $\{\mu_2/\mu_1, N\}$-plane, which happens at $\mathfrak{V}_1 = S_2$. In that case equation (1) has one root for every value of N and μ_2/μ_1.

b) $\mathfrak{V}_2 > \mathfrak{V}_1 > S_2 \, (1 > \omega > S_2/\mathfrak{V}_2)$.

The RAYLEIGH-root ζ_R, being smaller than 1, satisfies equation (1); as this is henceforward always so, there is always at least one root of equation (1) at $N = \sim$. The STONELEY-root exists, if $\mu_2/\mu_1 < a_1$ and also if $\mu_2/\mu_1 > a_2$. Therefore equation (1) has one root (namely ζ_R) at $N = \sim$, if $a_1 < \mu_2/\mu_1 < a_2$. Hence the part of the $\{\mu_2/\mu_1, N\}$-plane, which lies at large values of N between the two asymptotes $N = a_1$ and $N = a_2$, is the region where one root exists. In the remaining parts (the vertically shaded areas) two roots are possible, which become ζ_R and ζ_S if $N = \sim$. These parts increase with increasing value of ω (fig. 9—10).

Applied Mechanics. — *On the state of stress in perforated strips and plates.* (3d communication.) By K. J. SCHULZ. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of April 25, 1942.)

5. *The half-infinite plate and its periodic stress functions.* Another auxiliary problem, connected with the strip problem to be treated in section 7, is constituted by the half-infinite plate $z \leqq 0$, loaded at its boundary $z = 0$ by systems of normal and shearing stresses, which in the direction of the y-axis are periodical with the period b, which are symmetrical with respect to the z-axis, and the action of which vanishes at infinity. Such boundary systems are represented by

$$\sigma_{z\ bound.} = \sum_{n=1}^{\infty} i_n \cos 2\pi n y/b, \quad \tau_{yz\ bound.} = \sum_{n=1}^{\infty} D_n' \sin 2\pi n y/b. \quad . \quad (1)$$

and their corresponding stress function by

$$F' = \sum_{n=1}^{\infty} Z_n \cos 2\pi n \frac{y}{b},$$

where Z_n is supposed to depend solely on z[1]). If F' be substituted in the equation (2, 2), we obtain the following differential equation for Z_n

$$\left(\frac{d^2}{dz^2} - \frac{4\pi^2 n^2}{b^2}\right)^2 Z_n = 0 \quad (n \geqq 1)$$

the solution of which is given by

$$Z_n = A_n e^{2\pi n z/b} + B_n \,.\, 2\pi n \frac{z}{b} e^{2\pi n z/b} + C_n e^{-2\pi n z/b} + D_n \,.\, 2\pi n \frac{z}{b} e^{-2\pi n z/b} \ (n \geqq 1).$$

The constants C_n and D_n must be put equal to zero, in order to avoid stresses at infinity. Replacing y, z by η, ζ as defined in (4, 3) we therefore find:

$$F' = \sum_{n=1}^{\infty} (A_n e^{n\zeta} + B_n n \zeta e^{n\zeta}) \cos n\eta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The corresponding stresses follow from (4, 6) provided that in these formulae $U_{\tau 0}$ be replaced by F':

$$\left.\begin{aligned} \sigma_y &= +\frac{4\pi^2}{b^2} \sum_{n=1}^{\infty} n^2 [A_n + B_n (n\zeta + 2)] e^{n\zeta} \cos n\eta, \\ \sigma_z &= -\frac{4\pi^2}{b^2} \sum_{n=1}^{\infty} n^2 [A_n + B_n n\zeta] e^{n\zeta} \cos n\eta, \\ \tau_{yz} &= +\frac{4\pi^2}{b^2} \sum_{n=1}^{\infty} n^2 [A_n + B_n (n\zeta + 1)] e^{n\zeta} \sin n\eta. \end{aligned}\right\} \quad . \quad . \quad (3)$$

For $\zeta = 0$ these stresses must be identified with the stresses (1), from which it follows that:

$$A_n = -b^2 C_n'/4\pi^2 n^2, \ B_n = +b^2 (C_n' + D_n')/4\pi^2 n^2. \quad . \quad . \quad (4)$$

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1) Compare: L. N. G. FILON, Phil. Trans. Roy. Soc., A **201**, 63—155, (1903).

Consequently the solution of our problem is represented by

$$F' = +\frac{b^2}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(n\zeta-1)\,C_n' + n\zeta\,D_n'\right]e^{n\zeta}\cos n\eta \quad . \quad . \quad . \quad (5)$$

and

$$\left.\begin{aligned}\sigma_y &= +\sum_{n=1}^{\infty}\left[(n\zeta+1)\,C_n' + (n\zeta+2)\,D_n'\right]e^{n\zeta}\cos n\eta,\\ \sigma_z &= -\sum_{n=1}^{\infty}\left[(n\zeta-1)\,C_n' + n\zeta\,D_n'\right]e^{n\zeta}\cos n\eta,\\ \tau_{yz} &= +\sum_{n=1}^{\infty}\left[n\zeta\,C_n' + (n\zeta+1)\,D_n'\right]e^{n\zeta}\sin n\eta.\end{aligned}\right\} \quad . \; . \quad (6)$$

The boundary stress σ_y $(z=\zeta=0)$ in particular is given by

$$\sigma_{y\,bound.} = \sum_{n=1}^{\infty}(C_n' + 2D_n')\cos 2\pi n y/b. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In connection with the strip problem, to be treated in section 7, it is necessary to calculate the stresses in the points of the line $z=-2c$, *i.e.* for $\zeta=-4\pi\lambda/\mu$ (see **3**, 18 and **4**, 17). With the abbreviations

$$p_n' = \left(4\pi n\frac{\lambda}{\mu}+1\right)e^{-4\pi n\frac{\lambda}{\mu}},\ q_n' = r_n' = 4\pi n\frac{\lambda}{\mu}e^{-4\pi n\frac{\lambda}{\mu}},\ t_n' = \left(4\pi n\frac{\lambda}{\mu}-1\right)e^{-4\pi n\frac{\lambda}{\mu}}\ (n\geqq 1) \quad (8)$$

and by making use of (6), the result can be brought into the form

$$\left.\begin{aligned}\sigma_y &= \sum_{n=1}^{\infty}\left[C_n'(p_n'-2r_n') + D_n'(q_n'-2t_n')\right]\cos 2\pi n\frac{y}{b},\\ \sigma_z &= \sum_{n=1}^{\infty}(C_n'p_n' + D_n'q_n')\cos 2\pi n\frac{y}{b},\\ \tau_{yz} &= -\sum_{n=1}^{\infty}(C_n'r_n' + D_n't_n')\sin 2\pi n\frac{y}{b}.\end{aligned}\right\} \quad . \; . \quad (9)$$

If the half-infinite plate should cover the half-plane $z\geqq 0$, and if it should be loaded at its boundary by the stresses

$$\sigma_{z\,bound.} = \sum_{n=1}^{\infty}C_n'\cos 2\pi n\frac{y}{b},\ \tau_{yz\,bound.} = -\sum_{n=1}^{\infty}D_n'\sin 2\pi n\frac{y}{b}, \quad . \quad (10)$$

which are identical with the image of the stresses (1) with respect to the y-axis, it is readily seen that the stresses in the points of the line $z=+2c$ are given by

$$\left.\begin{aligned}\sigma_y &= \sum_{n=1}^{\infty}\left[C_n'(p_n'-2r_n') + D_n'(q_n'-2t_n')\right]\cos 2\pi n\frac{y}{b},\\ \sigma_z &= \sum_{n=1}^{\infty}(C_n'p_n' + D_n'q_n')\cos 2\pi n\frac{y}{b},\\ \tau_{yz} &= \sum_{n=1}^{\infty}(C_n'r_n' + D_n't_n')\sin 2\pi n\frac{y}{b}.\end{aligned}\right\} \quad . \; . \quad (11)$$

The boundary-stress σ_y $(z=0)$ is again given by (7).

6. *The stresses corresponding to F' in polar coordinates.* The last auxiliary problem, which has to be solved before the strip problem itself can be attacked, consists in determining the stress components — caused by F' — with respect to a system of polar coordinates, the origin of which has the coordinates $y=0$, $z=-c$, and the axis of which is parallel to the y-axis; especially the stress components are wanted in the points of the boundary $r=a$.

We consider the elementary stress functions

$$F_s'=e^{s\zeta}\cos s\eta,\quad F_s'^{*}=s\zeta e^{s\zeta}\cos s\eta\ (s\geqq 1),\quad .\ .\ .\ .\ .\quad (1)$$

of which the function (5, 5) is made up, and rewrite them as follows

$$F_s'=\Re e\, e^{-is\xi},\quad F_s'^{*}=s\zeta\,\Re e\, e^{-is\xi}\ (s\geqq 1).\quad .\ .\ .\ .\quad (2)$$

Then we perform a transition to an auxiliary system of rectangular coordinates y z, the origin of which coincides with the point $y=0$ $z=-c$, by replacing ζ by $\zeta-2\pi c/b=\zeta-2\pi\lambda/\mu$ and ξ by $\xi-2\pi c\,i/b=\xi-2\lambda\, i/\mu$. This transforms F_s' and $F_s'^{*}$ into

$$F_s'=e^{-2\pi s\frac{\lambda}{\mu}}\Re e\, e^{-is\xi},\ F_s'^{*}=\left(2\pi s\frac{r}{b}\sin\varphi-2\pi s\frac{\lambda}{\mu}\right)e^{-2\pi s\frac{\lambda}{\mu}}\Re e\, e^{-is\xi}\ (s\geqq 1)\quad (3)$$

Furthermore we replace $\Re e\, e^{-is\xi}$ by the series

$$\Re e\, e^{-is\xi}\Re e\left[1+\sum_{n=1}^{\infty}\frac{(-is\xi)^n}{n!}\right]=\Re e\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{s^{2n}\xi^{2n}}{2n!}-i\sum_{n=0}^{\infty}(-1)^n\frac{s^{2n+1}\xi^{2n+1}}{(2n+1)!}\right]$$

and finally we pass on to the polar system by putting $\xi=2\pi\frac{r}{b}e^{i\varphi}$, so that

$$\Re e\, e^{-is\xi}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi sr)^{2n}}{(2n)!\,b^{2n}}\cos 2n\varphi+\sum_{n=0}^{\infty}(-1)^n\frac{(2\pi sr)^{2n+1}}{(2n+1)!\,b^{2n+1}}\sin(2n+1)\varphi.$$

Substituting this expression into (3) we find

$$\left.\begin{aligned}
F_s'&=e^{-2\pi s\frac{\lambda}{\mu}}\left[\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi sr)^{2n}}{(2n)!\,b^{2n}}\cos 2n\varphi+\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi sr)^{2n+1}}{(2n+1)!\,b^{2n+1}}\sin(2n+1)\varphi\right],\\
F_s'^{*}&=e^{-2\pi s\frac{\lambda}{\mu}}\left[2\pi^2s^2\frac{r^2}{b^2}+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi sr)^{2n}}{(2n)!\,b^{2n}}\left(\frac{4\pi^2s^2}{2n+1}\frac{r^2}{b^2}+2n-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.\\
&\left.+2\pi^3s^3\frac{r^3}{b^3}\sin\varphi+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi sr)^{2n+1}}{(2n+1)!\,b^{2n+1}}\left(\frac{4\pi^2s^2}{2n+2}\frac{r^2}{b^2}+2n+1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right]\ (s\leqq 1).
\end{aligned}\right\}\quad (4)$$

In order to proceed in a way formally analogous to that followed in sections 3 and 4, we introduce — with respect to the original system of coordinates — the new stress functions $F_{\sigma s}'$ and $F_{\tau s}'$, the first of which produces the boundary stresses $\sigma_z=\cos 2\pi s\, y/b$, $\tau_{yz}=0$, and the second one the stresses $\sigma_z=0$, $\tau_{yz}=\sin 2\pi s\, y/b$.

It follows from (5, 5) that — with respect to the original system of coordinates — the new functions are defined by

$$F_{\sigma s}'=\frac{b^2}{4\pi^2s^2}(s\zeta-1)e^{s\zeta}\cos s\eta,\ F_{\tau s}'=\frac{b^2}{4\pi^2s^2}s\zeta e^{s\zeta}\cos s\eta\quad (s\geqq 1),\quad .\quad (5)$$

and therefore can be expressed in terms of F_s' and $F_s'^{*}$ as follows:

$$F_{\sigma s}'=\frac{b^2}{4\pi^2s^2}(F_s'^{*}-F_s'),\quad F_{\tau s}'=\frac{b^2}{4\pi^2s^2}F_s'^{*}\quad (s\geqq 1)$$

If now the expressions (4) are substituted into these formulae, we get the required Fourier series for $F'_{\sigma s}$ and $F'_{\tau s}$:

$$F'_{\sigma s}=e^{-2\pi s\frac{\lambda}{\mu}}\left[\tfrac{1}{2}r^2+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}r^{2n}}{(2n)!\,b^{2n-2}}\left(\frac{4\pi^2s^2}{2n+1}\frac{r^2}{b^2}+2n-2-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+\tfrac{1}{2}\pi s\frac{r^3}{b}\sin\varphi+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}r^{2n+1}}{(2n+1)!\,b^{2n-1}}\left(\frac{4\pi^2s^2}{2n+2}\frac{r^2}{b^2}+2n-1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right],$$

$$F'_{\tau s}=e^{-2\pi s\frac{\lambda}{\mu}}\left[\tfrac{1}{2}r^2+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}r^{2n}}{(2n)!\,b^{2n-2}}\left(\frac{4\pi^2s^2}{2n+1}\frac{r^2}{b^2}+2n-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+\tfrac{1}{2}\pi s\frac{r^3}{b}\sin\varphi+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}r^{2n+1}}{(2n+1)!\,b^{2n-1}}\left(\frac{4\pi^2s^2}{2n+2}\frac{r^2}{b^2}+2n+1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right].$$

The stresses corresponding to $F'_{\sigma s}$ $(s\geqq 1)$ are given by — see (2, 4) —

$$\sigma_r=e^{-2\pi s\frac{\lambda}{\mu}}\left[1-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2(2n-2)}{2n(2n-1)}\frac{r^2}{b^2}+2n-2-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+\pi s\frac{r}{b}\sin\varphi-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2(2n-1)}{2n(2n+1)}\frac{r^2}{b^2}+2n-1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right],$$

$$\sigma_\varphi=e^{-2\pi s\frac{\lambda}{\mu}}\left[1+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2(2n+2)}{2n(2n-1)}\frac{r^2}{b^2}+2n-2-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+3\pi s\frac{r}{b}\sin\varphi+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2(2n+3)}{2n(2n+1)}\frac{r^2}{b^2}+2n-1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right],$$

$$\tau_{r\varphi}=e^{-2\pi s\frac{\lambda}{\mu}}\left[+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2}{2n-1}\frac{r^2}{b^2}+2n-2-4\pi s\frac{\lambda}{\mu}\right)\sin 2n\varphi-\right.$$
$$\left.-\pi s\frac{r}{b}\cos\varphi-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2}{2n}\frac{r^2}{b^2}+2n-1-4\pi s\frac{\lambda}{\mu}\right)\cos(2n+1)\varphi\right]$$

those, corresponding to $F'_{\tau s}$ $(s\geqq 1)$ by:

$$\sigma_r=e^{-2\pi s\frac{\lambda}{\mu}}\left[1-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2(2n-2)}{2n(2n-1)}\frac{r^2}{b^2}+2n-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+\pi s\frac{r}{b}\sin\varphi-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2(2n-1)}{2n(2n+1)}\frac{r^2}{b^2}+2n+1-4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right],$$

$$\sigma_\varphi=e^{-2\pi s\frac{\lambda}{\mu}}\left[1+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2(2n+2)}{2n(2n-1)}\frac{r^2}{b^2}+2n-4\pi s\frac{\lambda}{\mu}\right)\cos 2n\varphi+\right.$$
$$\left.+3\pi s\frac{r}{b}\sin\varphi+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2(2n+3)}{2n(2n+1)}\frac{r^2}{b^2}+2n+1+4\pi s\frac{\lambda}{\mu}\right)\sin(2n+1)\varphi\right],$$

$$\tau_{r\varphi}=e^{-2\pi s\frac{\lambda}{\mu}}\left[+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-2}}{(2n-2)!}\left(\frac{r}{b}\right)^{2n-2}\left(\frac{4\pi^2s^2}{2n-1}\frac{r^2}{b^2}+2n-4\pi s\frac{\lambda}{\mu}\right)\sin 2n\varphi-\right.$$
$$\left.-\pi s\frac{r}{b}\sin\varphi-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2\pi s)^{2n-1}}{(2n-1)!}\left(\frac{r}{b}\right)^{2n-1}\left(\frac{4\pi^2s^2}{2n}\frac{r^2}{b^2}+2n+1-4\pi s\frac{\lambda}{\mu}\right)\cos(2n+1)\varphi\right].$$

Restricting ourselves to points of the circle $r = a$, we find

$$\left.\begin{aligned}
\sigma_r &= \tfrac{1}{2}\left[p_0^s + \sum_{n=1}^{\infty} p_{2n}^s \cos 2n\varphi + \sum_{n=0}^{\infty} p_{2n+1}^s \sin(2n+1)\varphi\right],\\
\sigma_\vartheta &= \tfrac{1}{2}\left[p_0^s + \sum_{n=1}^{\infty} (p_{2n}^s + 2r_{2n}^s)\cos 2n\varphi + \sum_{n=0}^{\infty} (p_{2n+1}^s + 2r_{2n+1}^s)\sin(2n+1)\varphi\right],\\
\tau_{r\vartheta} &= \tfrac{1}{2}\left[\sum_{n=1}^{\infty} r_{2n}^s \sin 2n\varphi - \sum_{n=0}^{\infty} r_{2n+1}^s \cos(2n+1)\varphi\right],
\end{aligned}\right\} F'_{\sigma s}\ (s \geqq 1),$$

$$\left.\begin{aligned}
\sigma_r &= \tfrac{1}{2}\left[q_0^s + \sum_{n=1}^{\infty} q_{2n}^s \cos 2n\varphi + \sum_{n=0}^{\infty} q_{2n+1}^s \sin(2n+1)\varphi\right],\\
\sigma_\vartheta &= \tfrac{1}{2}\left[q_0^s + \sum_{n=1}^{\infty} (q_{2n}^s + 2t_{2n}^s)\cos 2n\varphi + \sum_{n=0}^{\infty} q_{2n+1}^s + 2t_{2n+1}^s)\sin(2n+1)\varphi\right],\\
\tau_{r\vartheta} &= \tfrac{1}{2}\left[\sum_{n=1}^{\infty} t_{2n}^s \sin 2n\varphi - \sum_{n=0}^{\infty} t_{2n+1}^s \cos(2n+1)\varphi\right]
\end{aligned}\right\} F'_{\tau s}\ (s \geqq 1), \qquad (9)$$

with

$$\left.\begin{aligned}
p_0^s &= 2e^{-2\pi s\frac{\lambda}{\mu}},\ p_1^s = 2\pi s\lambda e^{-2\pi s\frac{\lambda}{\mu}},\ p_n^s = -(-1)^{\frac{n}{2},\frac{n-1}{2}} e^{-2\pi s\frac{\lambda}{\mu}} \frac{(2\pi s\lambda)^{n-2}}{n-2!}\left(\frac{4\pi^2 s^2\lambda^2(n-2)}{n(n-1)} + n - 2 - 4\pi s\frac{\lambda}{\mu}\right),\\
q_0^s &= 2e^{-2\pi s\frac{\lambda}{\mu}},\ q_1^s = 2\pi s\lambda e^{-2\pi s\frac{\lambda}{\mu}},\ q_n^s = -(-1)^{\frac{n}{2},\frac{n-1}{2}} e^{-2\pi s\frac{\lambda}{\mu}} \frac{(2\pi s\lambda)^{n-2}}{(n-2)!}\left(\frac{4\pi^2 s^2\lambda^2(n-2)}{n(n-1)} + n - 4\pi s\frac{\lambda}{\mu}\right),\\
r_1^s &= 2\pi s\lambda e^{-2\pi s\frac{\lambda}{\mu}},\ r_n^s = +(-1)^{\frac{n}{2},\frac{n-1}{2}} e^{-2\pi s\frac{\lambda}{\mu}} \frac{(2\pi s\lambda)^{n-2}}{(n-2)!}\left(\frac{4\pi^2 s^2\lambda^2}{n-1} + n - 2 - 4\pi s\frac{\lambda}{\mu}\right),\\
t_1^s &= 2\pi s\lambda e^{-2\pi s\frac{\lambda}{\mu}},\ t_n^s = +(-1)^{\frac{n}{2},\frac{n-1}{2}} e^{-2\pi s\frac{\lambda}{\mu}} \frac{(2\pi s\lambda)^{n-2}}{(n-2)!}\left(\frac{4\pi^2 s^2\lambda^2}{n-1} + n - 4\pi s\frac{\lambda}{\mu}\right)\ (n \geqq 2).
\end{aligned}\right\} \qquad (10)$$

The factor $(-1)^{\frac{n}{2},\frac{n-1}{2}}$ is to be understood in the same sense as the factor $(-1)^{\frac{s}{2},\frac{s-1}{2}}$ at the end of section 4.

If it should be required to calculate the stresses corresponding to the function

$$F' = \sum_{s=1}^{\infty}(C'_s F'_{\sigma s} + D'_s F'_{\tau s}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the result can be given by using 7 and 8, or respectively (9) and (10). It may be superfluous to write down the general formulae. As to the convergence of the series, used in this section it will easily be seen that the series (7) and (8) converge for all values of r/b.

7. The symmetrical strip with one row of holes under tension.

The obtained results will now be used to determine the state of stress in a symmetrical strip (width $= 2c$) with one row of holes (radius a, pitch b), subjected to pure tension in the longitudinal (y-) direction. To this end we consider an infinite plate on which the circular boundaries of the holes (later to be made), and the upper and lower straight boundaries of the strip itself may be indicated, and distinguished by the figures I, II, III. This plate is subjected: (a) to a uniform tension p' in the direction of the y-axis, (b) to

the action of a stress function (**3**, 20) related to the centres of the holes, (*c*) to the action of two stress-functions (**6**, 11) one of which is related to a system of coordinates connected with the upper straight boundary of the strip, the other one to a system of coordinates connected with its lower straight boundary (see for the definition of these coordinate systems section **6**). If the free constants of the different stress functions are chosen in such a way that along the boundaries I, II, III of the strip neither normal nor tangential stresses occur, the material outside the proper strip may be removed and its real state of stress and strain will be obtained.

Obviously the symmetry of the problem requires that in the stress function F (**3**, 20) those terms, which are provided with the constants $\overline{C}_{2s+1}$, $\overline{D}_{2s+1}$ $(s \geqq 0)$ be suppressed. The stresses arising from the remaining stress function

$$F = C_0 U_{\sigma 0} + \sum_{s=1}^{\infty} (C_{2s} U_{\sigma, 2s} + D_{2s} U_{\tau, 2s}) \quad . \quad . \quad . \quad . \quad (1)$$

are represented by

$$\left.\begin{aligned}
\sigma_r &= C_0 + \sum_{n=1}^{\infty} C_{2n} \cos 2n\varphi + C_0 (h_0^0 + \sum_{n=1}^{\infty} h_{2n}^0 \cos 2n\varphi) + \\
&\quad + \sum_{s=1}^{\infty} (C_{2s} h_0^{2s} + D_{2s} i_0^{2s}) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} (C_{2s} h_{2n}^{2s} + D_{2s} i_{2n}^{2s}) \cos 2n\varphi, \\
\tau_{r\varphi} &= \sum_{n=1}^{\infty} D_{2n} \sin 2n\varphi + C_0 \sum_{n=1}^{\infty} j_{2n}^0 \sin 2n\varphi + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} (C_{2s} j_{2n}^{2s} + D_{2s} k_{2n}^{2s}) \sin 2n\varphi. \\
\sigma_\varphi &= -C_0 + \sum_{n=1}^{\infty} (C_{2n} - 2D_{2n}) \cos 2n\varphi + C_0 [h_0^0 + \sum_{s=1}^{\infty} (h_{2n}^0 + 2j_{2n}^0) \cos 2n\varphi] + \\
&\quad + \sum_{s=1}^{\infty} (C_{2s} h_0^{2s} + D_{2s} i_0^{2s}) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} [C_{2s} (h_{2n}^{2s} + 2j_{2n}^{2s}) + D_{2s} (i_{2n}^{2s} + 2k_{2n}^{2s})] \cos 2n\varphi
\end{aligned}\right\} \quad (2)$$

as far as the points of the hole-boundaries I are concerned (comp. **3**, 15), and by

$$\left.\begin{aligned}
\sigma_z &= C_0 \sum_{n=1}^{\infty} h_n'^0 \cos 2\pi n \frac{y}{b} + \sum_{n=1}^{\infty} (C_{2s} h_n'^{2s} + D_{2s} i_n'^{2s}) \cos 2\pi n \frac{y}{b}, \\
\tau_{yz} &= \pm C_0 \sum_{n=1}^{\infty} j_n'^0 \sin 2\pi n \frac{y}{b} \pm \sum_{n=1}^{\infty} (C_{2s} j_n'^{2s} + D_{2s} k_n'^{2s}) \sin 2\pi n \frac{y}{b}, \\
\sigma_y &= C_0 \sum_{n=1}^{\infty} (h_n'^0 - 2j_n'^0) \cos 2\pi n \frac{y}{b} + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} [(C_{2s} (h_n'^{2s} - 2j_n'^{2s}) + \\
&\quad + D_{2s} (i_n'^{2s} - 2k_n'^{2s})] \cos 2\pi n \frac{y}{b},
\end{aligned}\right\} \quad (3)$$

for the points of the boundaries II and III. In the latter formulae (3) the upper one of the ambiguous signs relates to the boundary II, the lower one to the boundary III.

The two stress functions (**6**, 11), just now mentioned under (*c*) and connected with the boundaries II and III, produce stresses in the points of their "own" boundary which for the boundary II can be derived from (**5**, 6) by putting $\zeta = 0$; for it will be seen with a view on (**6**, 5) that the stress-function (**5**, 5) from which the stresses (**5**, 6) are deduced is identical with the function (4).

The stresses at the boundary III are expressed by the same formulae as far as σ_y and σ_z are concerned; τ_{yz} on the contrary has the opposite sign. Taking these remarks into

account the two sets of stress-components, produced by the stress-functions (4) at their *own* boundaries, are represented by

$$\left.\begin{aligned} \sigma_z &= \sum_{n=1}^{\infty} C'_n \cos 2\pi n \frac{y}{b}, \\ \tau_{yz} &= \pm \sum_{n=1}^{\infty} D'_n \sin 2\pi n \frac{y}{b}, \\ \sigma_y &= \sum_{n=1}^{\infty} (C'_n + 2D'_n) \cos 2\pi n \frac{y}{b}. \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

By the formulae (5, 11) and (5, 9) we find the stresses to which the stress functions (4) give rise at their *opposite* boundaries. The first set relates to the stresses at the boundary II (caused by the stress-function connected with the boundary III); the second set gives the stresses at the boundary III (caused by the stress-function, connected with the boundary II). The result is represented by the single set of stresses (6)

$$\left.\begin{aligned} \sigma_z &= \sum_{n=1}^{\infty} (C'_n p'_n + D'_n q'_n) \cos 2\pi n \frac{y}{b}, \\ \tau_{yz} &= \pm \sum_{n=1}^{\infty} (C'_n r'_n + D'_n t'_n) \sin 2\pi n \frac{y}{b}, \\ \sigma_y &= \sum_{n=1}^{\infty} [C'_n (p'_n - 2r'_n) + D'_n (q'_n - 2t'_n) \cos 2\pi n \frac{y}{b}. \end{aligned}\right\} \quad \ldots \quad (6)$$

The boundary II requires the + sign, the boundary III the — sign in the middle formula.

Finally the stresses at the hole-boundaries I, which are due to the joint action of the stress functions (4) must be calculated. They can be deduced from (6, 9) if only all terms with even index n are doubled, and all terms with odd index n are suppressed. We get therefore

$$\left.\begin{aligned} \sigma_r &= \sum_{s=1}^{\infty} (C'_s p_0^s + D'_s q_0^s) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} (C'_s p_{2n}^s + D'_s q_{2n}^s) \cos 2n\varphi, \\ \tau_{r\varphi} &= \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} (C'_s \tau_{2n}^s + D'_s t_{2n}^s) \sin 2n\varphi, \\ \sigma_\varphi &= \sum_{s=1}^{\infty} (C'_s p_0^s + D'_s q_0^s) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} [C'_s (p_{2n}^s + 2r_{2n}^s) + \\ &\qquad + D'_s (q_{2n}^s + 2t_{2n}^s)] \cos 2n\varphi. \end{aligned}\right\} \quad (7)$$

It now remains to describe the state of stress mentioned beforehand under (*a*), which in contrast with the other ones is well-known. The stresses occurring at the hole-boundaries I will be designed by $\sigma_r^{(0)}$, $\tau_{r\varphi}^{(0)}$, $\sigma_\varphi^{(0)}$; those occurring at the boundaries II and III by $\sigma_z^{(0)}$, $\tau_{yz}^{(0)}$, $\sigma_\varphi^{(0)}$. For the sake of generality their FOURIER-series as far as $\sigma_r^{(0)}$, $\tau_{r\varphi}^{(0)}$, $\sigma_z^{(0)}$, $\tau_{yz}^{(0)}$ are concerned, will be written as:

$$\sigma_r^{(0)} = -C_0^{(0)} - \sum_{n=1}^{\infty} C_{2n}^{(0)} \cos 2n\varphi, \qquad \tau_{r\varphi}^{(0)} = -\sum_{n=1}^{\infty} D_{2n}^{(0)} \sin 2n\varphi, \quad . \quad (8)$$

respectively

$$\sigma_z^{(0)} = -\sum_{n=1}^{\infty} C_n'^{(0)} \cos 2\pi n \frac{y}{b}, \qquad \tau_{yz}^{(0)} = \mp \sum_{n=1}^{\infty} D_n'^{(0)} \sin 2\pi n \frac{y}{b} \quad . \quad . \quad (9)$$

(compare the formulae (11) and (13) which give the actual values of the coefficients C and D).

It has already been stated that the requirement of our problem consists in the disappearance of all normal and tangential boundary stresses. Therefore we have to sum up the corresponding stresses (2), (7), and (8) respectively (3), (5), (6) and (9) and to put the resulting stresses $\sigma_r, \tau_{r\varphi}$, respectively σ_z, τ_{yz} equal to zero. This leads to the following infinite system of linear equations for the unknown coefficients

$$C_{2n}\,(n \geqq 0),\ D_{2n}\,(n \geqq 1),\ C'_n\,(n \geqq 1),\ D'_n\,(n \geqq 1)$$

$$\left.\begin{aligned}
C_{2n} &= C_{2n}^{(0)} - [C_0\, h_{2n}^0 + \sum_{s=1}^{\infty} (C_{2s}\, h_{2n}^{2s} + D_{2s}\, i_{2n}^{2s}) + \sum_{s=1}^{\infty} (C'_s\, p_{2n}^s + D'_s\, q_{2n}^s)] && (n \geqq 0),\\
D_{2n} &= D_{2n}^{(0)} - [C_0\, j_{2n}^0 + \sum_{s=1}^{\infty} (C_{2s}\, j_{2n}^{2s} + D_{2s}\, k_{2n}^{2s}) + \sum_{s=1}^{\infty} (C'_s\, r_{2n}^s + D'_s\, t_{2n}^s)] && (n \geqq 1),\\
C'_n &= C_n^{'(0)} - [C_0\, h_n^{'0} + \sum_{s=1}^{\infty} (C_{2s}\, h_n^{'2s} + D_{2s}\, i_n^{'2s}) + C'_n\, p'_n + D'_n\, q'_n] && (n \geqq 1),\\
D'_n &= D_n^{'(0)} - [C_0\, j_n^{'0} + \sum_{s=1}^{\infty} (C_{2s}\, j_n^{'2s} + D_{2s}\, k_n^{'2s}) + C'_n\, r'_n + D'_n\, t'_n] && (n \geqq 1).
\end{aligned}\right\} \quad (10)$$

The quantities h, i, j, k, p, q, r, t design numerical coefficients, the computation of which has been described in the sections **3**—**6**. As to the coefficients $C_{2n}^{(0)}$, $D_{2n}^{(0)}$, $C_n^{'(0)}$, $D_n^{'(0)}$: it is seen at once that the state of stress, caused by the uniform tension p' is described by $(\sigma_y^0) = p'$, $\sigma_z^{(0)} = 0$, $\tau_{yz}^{(0)} = 0$, so that

$$C_n^{'(0)} = D_n^{'(0)} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In polar coordinates the same state of stress is given by

$$\sigma_r^{(0)} = \tfrac{1}{2}\, p'\,(1 + \cos 2\varphi), \quad \sigma_\varphi^{(0)} = \tfrac{1}{2}\, p'\,(1 - \cos 2\varphi), \quad \tau_{r\varphi}^{(0)} = -\tfrac{1}{2}\, p' \sin 2\varphi \quad (12)$$

from which it follows, that

$$C_0^{(0)} = -\tfrac{1}{2}\, p', \quad C_2^{(0)} = -\tfrac{1}{2}\, p', \quad D_2^{(0)} = +\tfrac{1}{2}\, p', \quad C_{2n}^{(0)} = D_{2n}^{(0)} = 0 \quad n > 1. \quad (13)$$

The solution of the system (10) will be pursued in an iterative way (comp. "A" **6**, **8**), the precept of which will be described now.

(To be continued.)

ERRATUM:

On p. 462 the last paragraph but one is to be read as follows:
The two stress functions (**6**, 11),

$$F' = \sum_{s=1}^{\infty} (C'_s\, F'_{\tau s} + D'_s\, F'_{\tau s}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

just now mentioned under (c) and connected with the boundaries II and III, produce stresses in the points of their "own" boundary which for the boundary II can be derived from (**5**, 6) by putting $\zeta = 0$; for it will be seen with a view on (**6**, 5) that the stress-function (**5**, 5) from which the stresses (**5**, 6) are deduced is identical with the function (4).

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Elfte Mitteilung.) Von W. J. BOS. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of April 25, 1942.)

In der neunten Mitteilung begegneten wir dem System der ∞^2 Ebenen, welche vier aufeinanderfolgende Erzeugenden schneiden. Wir fanden dort, dass $K=0$ (244) die Gleichung in Raumkoordinaten der Mannigfaltigkeit dieser „Vierpunktebenen" darstellt. Es gibt, wie wir weiter zeigten, zwei Büschel von Vierpunktebenen durch den Heftpunkt H: die Vierpunktebenen A und B (§ 27); jedes dieser Büschel enthält eine Fünfpunktebene. ((248), (249)).

In dieser Mitteilung beginnen wir met der allgemeine Untersuchung der Vier-, Fünf- und Sechspunktebenen; d.h. der Ebenen, welche eine bestimmte Erzeugende 0_{ik} von F schneiden, und in diesen Schnittpunkten mit F einen Kontakt resp. dritter, vierter und fünfter Ordnung haben.

§ 31.

Wir gehen aus von den fünf linearen Räumen mit den Gleichungen:

$$x_{02}=0,\ x_{03}=0,\ x_{04}=0,\ x_{22}=0,\ x_{23}=0 \quad . \quad . \quad . \quad . \quad (270)$$

Diese Räume sind, wenigstens wenn $Q \not\equiv 0$ ist, unabhängig; denn

$$\left.\begin{aligned}(M'_{02}, M'_{03}, M'_{04}, M'_{22}, M'_{23}) &= \frac{1}{4!}\Sigma(0^2 2^2)_{iklm}(M'_{03}, M'_{04}, M'_{22}, M'_{23})_{iklm} = \\ &= \frac{1}{4!}\begin{vmatrix} 0 & 0 & 0_{22} & 0_{23} \\ 0 & 0 & 0_{22} & 0_{23} \\ 2_{03} & 2_{04} & 0 & 2_{23} \\ 2_{03} & 2_{04} & 0 & 2_{23} \end{vmatrix} = \frac{1}{6}\cdot 0_{22,23}\cdot 2_{03,\,04} = \frac{8}{9}\cdot Q^2 \neq 0\end{aligned}\right\} \quad (271)$$

Sind die fünf Räume (270) die Fundamentalräume eines Koordinatensystemes (X), dann lauten ihre Gleichungen: $X_i=0$ $(i=1, 2, 3, 4, 5.)$

Diese Räume schneiden einander in zehn Ebenen. Die Gleichungen dieser Ebenen sind:

$$\left.\begin{aligned} \Pi_{12}=\pi_{02,03}=0,\ \Pi_{13}=\pi_{02,04}=0,\ \Pi_{14}=\pi_{02,22}=0,\ \Pi_{15}=\pi_{02,23}=0, \\ \Pi_{23}=\pi_{03,04}=0,\ \Pi_{24}=\pi_{03,22}=0,\ \Pi_{25}=\pi_{03,23}=0, \\ \Pi_{34}=\pi_{04,22}=0,\ \Pi_{35}=\pi_{04,23}=0.\end{aligned}\right\} \quad (272)$$

$\Pi_{12}=0$ ist also z.B. die Gleichung der erste Ebene im Koordinatensysteme (X).

Die Gleichung jedes linearen Linienkomplexes im R_4, also auch die Gleichung jeder Ebene, können wir in der Gestalt schreiben:

$$\Phi=(\Phi^3\pi^2)=\Sigma c'_{ik}\,\Pi_{ik}=c'_{12}\pi_{02,03}+c'_{13}\pi_{02,04}+\ldots+c'_{45}\pi_{22,23}=0 \quad . \quad (273)$$

$\phi=0$ ist die Gleichung einer Ebene, wenn der Komplex speziell ist, also wenn:

$$(c'^2\,d'^2\,U')\equiv 0, \quad (d'_{ik}=c'_{ik})$$

Oder:

$$\left.\begin{aligned} c'_{23}\,c'_{45}+c'_{24}\,c'_{53}+c'_{25}\,c'_{34}&=0\\ c'_{13}\,c'_{45}+c'_{14}\,c'_{53}+c'_{15}\,c'_{34}&=0\\ c'_{12}\,c'_{45}+c'_{14}\,c'_{52}+c'_{15}\,c'_{24}&=0\\ c'_{12}\,c'_{35}+c'_{13}\,c'_{52}+c'_{15}\,c'_{23}&=0\\ c'_{12}\,c'_{34}+c'_{13}\,c'_{42}+c'_{14}\,c'_{23}&=0 \end{aligned}\right\} \quad \ldots\ldots \quad (274)$$

Wir können uns beschränken auf die Ebenen, welche die Erzeugende 0_{ik} schneiden, durch $c'_{45}=0$ zu nehmen, denn die ersten neun Ebenen (272) schneiden 0_{ik}.

Schreiben wir statt (273)

$$\left.\begin{aligned} \Phi=(\Phi^3\,\pi^2)=\lambda_1\,\pi_{02,03}+\lambda_2\,\pi_{02,04}+\lambda_3\,\pi_{02,22}+\lambda_4\,\pi_{02,23}+\\ +\lambda_5\,\pi_{03,04}+\lambda_6\,\pi_{03,22}+\lambda_7\,\pi_{03,23}+\\ +\lambda_8\,\pi_{04,22}+\lambda_9\,\pi_{04,23}=0, \end{aligned}\right\} \quad . \quad (275)$$

wobei also $c'_{12}=-c'_{21}=\lambda_1$ u.s.w. gesetzt ist, dann lauten die Bedingungen dass ϕ eine Ebene ist:

$$\left.\begin{aligned} \lambda_6\,\lambda_9-\lambda_7\,\lambda_8&=0\\ \lambda_3\,\lambda_9-\lambda_4\,\lambda_8&=0\\ \lambda_3\,\lambda_7-\lambda_4\,\lambda_6&=0\\ \lambda_1\,\lambda_9-\lambda_2\,\lambda_7+\lambda_4\,\lambda_5&=0\\ \lambda_1\,\lambda_8-\lambda_2\,\lambda_6+\lambda_3\,\lambda_5&=0 \end{aligned}\right\} \quad \ldots\ldots \quad (276)$$

Diese Gleichungen sind aber nicht unabhängig. Man sieht leicht, dass man, wenn $\lambda_8\neq 0$ ist, (276) ersetzen kann durch:

$$\lambda_7=\frac{\lambda_6\,\lambda_9}{\lambda_8} \quad \ldots\ldots\ldots\ldots \quad (277a)$$

$$\lambda_4=\frac{\lambda_3\,\lambda_9}{\lambda_8} \quad \ldots\ldots\ldots\ldots \quad (277b)$$

$$\lambda_1=\frac{\lambda_2\,\lambda_6-\lambda_3\,\lambda_5}{\lambda_8} \quad \ldots\ldots\ldots\ldots \quad (277c)$$

Betrachten wir die λ_i $(i=1,2,\ldots,9)$ als Koordinaten in einem R_8, dann werden dadurch die Ebenen im R_4, welche 0_{ik} schneiden, abgebildet auf die Punkte eines R_5^5 in diesem R_8, bestimmt durch die Gleichungen (276).

Wir erhalten die $(k+1)$-Punktebenen, wenn wir in (275) den λ_i ausser (276) noch die Bedingungen

$$(\Phi^3\,1^2)=0,\ (\Phi^3\,2^2)=0,\ (\Phi^3\,3^2)=0,\ \ldots,\ (\Phi^3\,k^2)=0$$

auflegen. Die ersten vier dieser Bedingungen lauten:

$$(\Phi^3\,1^2)=\lambda_7\,.\,1_{03,23}+\lambda_8\,.\,1_{04,22}+\lambda_9\,.\,1_{04,23}=-Q\,.\,\lambda_7+\tfrac{8}{3}\,.\,Q\,.\,\lambda_8=0.$$

Oder:

$$3\,.\,\lambda_7=8\,.\,\lambda_8 \quad \ldots\ldots\ldots\ldots \quad (\Phi_1)$$

Ebenso folgt aus $(\phi^3 2^2) = 0$, wegen (1), (103) und (112):

$$\tfrac{2}{3} \cdot Q \cdot \lambda_4 - 4 \cdot Q \cdot \lambda_5 + \tfrac{1}{3} \cdot R \cdot \lambda_7 - \tfrac{2}{3} \cdot S \cdot \lambda_9 = 0 \quad . \quad . \quad . \quad . \quad (\phi_2)$$

Aus $(\phi^3 3^2) = 0$ finden wir nach (103), (110) und (41):

$$4 \cdot Q \cdot \lambda_2 - \tfrac{4}{3} \cdot Q \cdot \lambda_3 - \tfrac{1}{3} \cdot R \cdot \lambda_4 - \tfrac{2}{3} \cdot R \cdot \lambda_6 + \tfrac{1}{2} \cdot S \cdot \lambda_7 + \tfrac{4}{3} \cdot S \cdot \lambda_8 - \tfrac{1}{2} \cdot T \cdot \lambda_9 = 0 \ldots (\phi_3)$$

Weiter gibt $(\phi^3 4^2) = 0$ nach einiger Rechnung:

$$\left.\begin{aligned} -4 \cdot Q \cdot \lambda_1 + (\tfrac{10}{9} \cdot R - \tfrac{4}{3} \cdot Q') \lambda_3 + (\tfrac{2}{3} \cdot S_0 + \tfrac{2}{3} \cdot S + \tfrac{7}{9} \cdot R' - \tfrac{2}{3} \cdot Q'') \lambda_4 + \\ + (-\tfrac{2}{3} R' - 2 \cdot S) \lambda_6 + 4_{03,23} \cdot \lambda_7 + 4_{04,22} \cdot \lambda_8 + 4_{04,23} \cdot \lambda_9 = 0 \end{aligned}\right\} \quad (\phi_4)$$

§ 32.

Wir betrachten zuerst den Fall $\lambda_8 = 0$ und nehmen wieder $Q \neq 0$. Aus (ϕ_1) bekommen wir $\lambda_7 = 0$, und also wegen (276) die folgende Einteilung:

$$\lambda_8 = 0, \lambda_7 = 0 \left\{\begin{array}{ll} \text{(I)} & \lambda_9 = 0 \left\{\begin{array}{ll} (\text{I}^A) & \lambda_5 = 0,\ \lambda_6 = 0. \\ (\text{I}^B) & \lambda_4 = 0,\ \lambda_2 \cdot \lambda_6 - \lambda_3 \cdot \lambda_5 = 0 \end{array}\right. \\ \text{(II)} & \lambda_9 \neq 0,\ \lambda_3 = 0,\ \lambda_6 = 0,\ \lambda_1 \cdot \lambda_9 + \lambda_4 \cdot \lambda_5 = 0 \end{array}\right\} \quad . \quad . \quad (278)$$

Fall IA. Die ∞^3 Zweipunktebenen IA

$$(\Phi^3 \pi^2) = \lambda_1 \cdot \pi_{02,03} + \lambda_2 \cdot \pi_{02,04} + \lambda_3 \cdot \pi_{02,22} + \lambda_4 \cdot \pi_{02,23} = 0$$

sind die Ebenen im Tangentialraume.

Aus (ϕ_2) folgt: $\lambda_4 = 0$. Man sieht leicht: Die ∞^2 Dreipunktebenen IA sind die Ebenen durch H im Tangentialraume.

(ϕ_3) gibt hier: 3. $\lambda_2 = \lambda_3$. Für die Vierpunktebenen IA haben wir also die Darstellung:

$$(\Phi^3 \pi^2) = \lambda_1 \cdot \pi_{02,03} + \lambda_2 (\pi_{02,04} + 3 \cdot \pi_{02,22}) = 0 \quad . \quad . \quad . \quad . \quad (279)$$

Alle Ebenen (279) enthalten die Schnittgerade der Ebenen $\pi_{02,03} = 0$ und $\pi_{02,04} + 3 \cdot \pi_{02,22} = 0$. Mit Hilfe der Gleichungen (255) und (256) rechnet man leicht nach, dass diese Schnittgerade die Achse HG der Vierpunktebenen A ist.

Die Gleichung (279) gibt also die Vierpunktebenen A des § 27.

Mit (ϕ_4) finden wir dann weiter die Fünfpunktebene A (248) zurück.

Fall IB. Die ∞^3 Zweipunktebenen IB bestimmt durch

$$\left.\begin{aligned} (\Phi^3 \pi^2) = \lambda_1 \cdot \pi_{02,03} + \lambda_2 \cdot \pi_{02,04} + \lambda_3 \cdot \pi_{02,22} + \lambda_5 \cdot \pi_{03,04} + \lambda_6 \cdot \pi_{03,22} = 0 \\ \lambda_2 \cdot \lambda_6 - \lambda_3 \cdot \lambda_5 = 0 \end{aligned}\right\}$$

gehen durch H, liegen aber nicht in einem R_3.

(ϕ_2) gibt hier $\lambda_5 = 0$, also wegen (276) $\lambda_2 = 0$ oder $\lambda_6 = 0$.

$\lambda_6 = 0$ führt uns zurück zum Falle IA, während wir im Falle $\lambda_2 = 0$ haben:

$$(\Phi^3 \pi^3) = \lambda_1 \cdot \pi_{02,03} + \lambda_3 \pi_{02,22} + \lambda_6 \pi_{03,22} = 0.$$

Die drei Ebenen, woraus ϕ zusammengestellt ist, gehen durch die Gerade mit der Gleichung

$$\pi_{02,03,22} = 0$$

also, wegen (256), durch die Heftgerade a_{ik}.

Die ∞^2 Dreipunktebenen IB sind also die Ebenen durch die Heftgerade.

Aus (ϕ_3) folgt jetzt:

$$2 \,.\, Q \,.\, \lambda_2 + R \,.\, \lambda_6 = 0.$$

Wir finden die Vierpunktebenen B (§ 27) zurück in der Gestalt:

$$(\Phi^3 \pi^2) = \lambda \,.\, \pi_{02,03} + R \,.\, \pi_{02,22} - 2 \,.\, Q \,.\, \pi_{03,22} = 0 \quad . \quad . \quad . \quad (280)$$

Die Fünfpunktebene B (249) erhalten wir hieraus mit (ϕ_4).

Fall II. Hier haben wir ∞^3 Zweipunktebenen, bestimmt durch:

$$\left.\begin{array}{c} \Phi^3 \pi^2) = \lambda_1 \,.\, \pi_{02,03} + \lambda_2 \,.\, \pi_{02,04} + \lambda_4 \,.\, \pi_{02,23} + \lambda_5 \,.\, \pi_{03,04} + \lambda_9 \,.\, \pi_{04,23} = 0 \\ \lambda_1 \, \lambda_9 + \lambda_4 \, \lambda_5 = 0 \end{array}\right\} \quad (281)$$

ϕ ist eine Linearkombination von Ebenen, die alle durch den Punkt mit der Gleichung

$$(M'_{02}, M'_{03}, M'_{04}, M'_{23}, u') = 0$$

gehen. Ersetzt man M'_{02} durch $(0^2 2^2)$, so lässt sich dies schreiben als

$$\begin{vmatrix} 0 & 0 & 0_{23} & 0_{u'} \\ 0 & 0 & 0_{23} & 0_{u'} \\ 2_{03} & 2_{04} & 2_{23} & 2_{u'} \\ 2_{03} & 2_{04} & 2_{23} & 2_{u'} \end{vmatrix} = 4 \,.\, 2_{03,04} \,.\, 0_{23} \, 0_{u'} = -16 \,.\, Q \,.\, 0_{23} \, 0_{u'} = 0 \,. \quad . \quad (282)$$

Dieser Punkt gehört die Erzeugende 0_{ik} an und liegt in allen Ebenen ϕ (281). $0_{23} 0_{u'}$ ist aber keine Differentialkontravariante; die geometrischen Eigenschaften der Mehrpunktebenen II gelten also für jeden Punkt der Erzeugende 0_{ik}. So bekommen wir aus (281) mit (ϕ_2), (ϕ_3) und (ϕ_4):

Die Vierpunktebenen durch einen Punkt der Erzeugenden 0_{ik} bilden einen kwadratischen Hyperkegel. (Für den Punkt H ist dieser Hyperkegel ausgeartet in die zwei Ebenenbüschel A und B.)

Durch einen Punkt der Erzeugenden 0_{ik} gehen im Allgemeinen zwei Fünfpunktebenen. (Durch den Punkt H: die Fünfpunktebenen (248) und (249).)

Die letzte Behauptung wollen wir etwas näher untersuchen. Hierzu betrachten wir einen Punkt P von 0_{ik} mit der Gleichung:

$$P_{u'} = \mu \,.\, 0_{22} \, 0_{u'} + 0_{23} \, 0_{u'} = 0$$

P liegt in der Ebene ϕ (275) wenn:

$$\begin{aligned} (\mu \,.\, \lambda_4 \,.\, 0_{22,23} - \lambda_3 \,.\, 0_{22,23}) \, x_{02} + (\mu \,.\, \lambda_7 \,.\, 0_{22,23} - \lambda_6 \,.\, 0_{22,23}) \, x_{03} + \\ + (\mu \,.\, \lambda_9 \,.\, 0_{22,23} - \lambda_8 \,.\, 0_{22,23}) \, x_{04} \equiv 0 \, \{x\} \end{aligned}$$

Die Räume $x_{02} = 0$, $x_{03} = 0$, und $x_{04} = 0$ sind unabhängig; wir finden also:

$$\lambda_3 = \mu \,.\, \lambda_4; \; \lambda_6 = \mu \,.\, \lambda_7; \; \lambda_8 = \mu \,.\, \lambda_9 \quad . \quad . \quad . \quad . \quad . \quad (283)$$

Wenn wir noch die vierte Bedingung (276)

$$\lambda_1 \,.\, \lambda_9 = \lambda_2 \,.\, \lambda_7 - \lambda_4 \,.\, \lambda_5 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276d)$$

hinzufügen, dann sind die übrigen Beziehungen (276) von dieser Vierzahl abhängig. ϕ (275) ist also dann und nur dann eine Ebene wenn (283) und (276d) erfüllt sind. Mit Hilfe dieser Bedingungen, zusammen mit (ϕ_1), (ϕ_2) und (ϕ_3) kann man die

$\lambda_1, \lambda_2, \lambda_3, \lambda_5, \lambda_7$, und λ_8, in λ_4 und λ_9 ausdrücken. Einsetzung dieser Ausdrücke in (ϕ_4) gibt die folgende kwadratische Gleichung für das Verhältnis von λ_4 und λ_9:

$$\left.\begin{aligned} &\tfrac{2}{3}.Q.\lambda_4^2+\{-\tfrac{4}{3}.Q.\mu^2+(\tfrac{5}{3}.R-\tfrac{4}{3}Q.')\mu+\tfrac{2}{3}.S_0+\tfrac{7}{9}.R'\}\lambda_4.\lambda_9+ \\ &\{-\tfrac{2}{3}.R.\mu^3+(-\tfrac{16}{9}.R'-\tfrac{8}{3}.S)\mu^2+ \\ &+(-\tfrac{1}{2}.T+\tfrac{8}{3}.4_{03,23}+4_{04,22})\mu+4_{04,23}\}\lambda_9^2=0. \end{aligned}\right\} \quad (284)$$

Die Diskriminante dieser Gleichung ist vom vierten Grade in μ und verschwindet nicht identisch, wegen $Q \not\equiv 0$.

Durch P gehen also im Allgemeinen zwei verschiedene Fünfpunktebenen und *es gibt höchstens vier Punkte auf* 0_{ik}, *wodurch nur eine Fünfpunktebene geht.*

§ 33.

Wir beweisen noch den Satz:

Nur der Tangentialraum und der Beiraum enthalten mehr dann eine Vierpunktebene, nämlich die Büschel A und B ((279) und (280)).

Beweis: Zwei Ebenen c und d, liegend in einem $R_3 u'$, können wir immer betrachten als Schnittebenen des Raumes u' mit zwei Räumen v' und w'. Die Gleichungen von c und d kann man also in der Gestalt schreiben:

$$(c'\,\pi)^2=(\pi\,u')\,(\pi\,v')=0$$
$$(d'\,\pi)^2=(\pi\,u')\,(\pi\,w')=0$$

Diese Ebenen sind Vierpunktebenen, wenn:

$$\left.\begin{aligned} &(c'\,i)^2=(i\,u')\,(i\,v')=0 \\ &(d'\,i)^2=(i\,u')\,(i\,w')=0 \end{aligned}\quad (i=0,1,2,3.)\right\} \quad . \quad . \quad . \quad . \quad . \quad (285)$$

Sei m die Schnittgerade der Ebenen c und d, dann hat m die Gleichung:

$$(m^2\,\pi^3)=2\,(m'\,\pi)^3=12\,\Sigma m'_{ikl}\,\pi_{ikl}=(\pi\,u')\,(\pi\,v')\,(\pi\,w')=0 \quad . \quad . \quad (286)$$

Damit finden wir

$$(0^2\,m^2\,x)=2\,(m'\,0)^2\,(m'\,x)=\tfrac{1}{3}(u'\,0)\,(v'\,0)\,.\,(w'\,x)-\tfrac{1}{3}(u'\,0)\,(w'\,0)\,.\,(v'\,x)+ \\ +\tfrac{1}{3}(v'\,0)\,(w'\,0)\,.\,(u'\,x)$$

Oder wegen (285):

$$(0^2\,m^2\,x)=\tfrac{1}{3}\,(v'\,0)\,(w'\,0)\,.\,(u'\,x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (287a)$$

Ebenso:

$$(1^2\,m^2\,x)=\tfrac{1}{3}\,(v'\,1)\,(w'\,1)\,.\,(u'\,x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (287b)$$
$$(2^2\,m^2\,x)=\tfrac{1}{3}\,(v'\,2)\,(w'\,2)\,.\,(u'\,x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (287c)$$
$$(3^2\,m^2\,x)=\tfrac{1}{3}\,(v'\,3)\,(w'\,3)\,.\,(u'\,x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (287d)$$

Wir behaupten nun:

Wenn der Raum mit Gleichung $(0^2\,m^2\,x)=0$ wirklich existiert, also wenn $(v'\,0)\,(w'\,0) \neq 0$ ist, dann können wir zeigen, dass die Schnittebene dieses Raumes mit dem Tangentialraume unbestimmt wird. Wegen (287*a*) liegen die Ebenen c und d dann im Tangentialraume.

Die Gleichung der genannten Schnittebene lautet nämlich:

$$\pi_{02,0m}=-\pi_{11,0m}=2\,.\,1_{1\pi,0m}=-2\,.\,0_{1\pi,1m}=0.$$

Wenn $(1^2 m^2 x) \equiv 0$ ist, haben wir in der Tat

$$\pi_{02,0m} = -2 \cdot 0_{1\pi,1m} \equiv 0.$$

Und wenn $(1^2 m^2 x) \not\equiv 0$ ist, also wenn $(v'1)(w'1) \neq 0$ ist, dann bekommen wir wegen (287*a*) und (287*b*):

$$(1^2 m^2 x) = \tfrac{1}{3}(v'1)(w'1) \cdot (u'x) = \frac{(v'1)(w'1)}{(v'0)(w'0)}(0^2 m^2 x).$$

Hiermit wird

$$\pi_{02,0m} = -2 \cdot 0_{1\pi,1m} = -2\frac{(v'1)(w'1)}{(v'0)(w'0)} 0_{1\pi,0m} \equiv 0.$$

Damit ist unsere Behauptung bewiesen.

Auf dieselbe Weise finden wir, dass im Falle $(0^2 m^2 x) \equiv 0$, aber $(1^2 m^2 x) \not\equiv 0$, die Schnittebene des Tangentialraumes mit dem Raume mit Gleichung

$$(1^2 m^2 x) = 0$$

unbestimmt wird, sodass die Vierpunktebenen c und d auch jetzt im Tangentialraume liegen.

So fortfahrend kann man Folgendes zeigen:

Die Annahme, es gäbe im Raume u' zwei Vierpunktebenen c und d, führt zu den zwei Möglichkeiten:

1. Der Raum u' ist der Tangentialraum; c und d sind Vierpunktebenen A.
2. $(0^2 m^2 x) \equiv 0$, $(1^2 m^2 x) \equiv 0$, $(2^2 m^2 x) \equiv 0$.

In diesem Falle schneidet m drei aufeinanderfolgende Erzeugenden; m ist also die Heftgerade a_{ik}. Die Vierpunktebenen durch a_{ik} sind die Ebenen $(\alpha_{2y})_{ikl}$ wofür $(\alpha^2 y 3^2) = 0$ ist, d.h., wenigstens wenn $Q \not\equiv 0$ ist, die Vierpunktebenen B im Beiraume. (Vgl. § 27.)

Damit ist der Satz bewiesen.

Mathematics. — *Sur le théorème de* MINKOWSKI, *concernant un système de formes linéaires réelles. III. Troisième communication: Démonstration des lemmes 5 et 6.* Par J. F. KOKSMA et B. MEULENBELD. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 25, 1942.)

§ 1. Dans cette communication nous démontrerons les lemmes 4 et 5, cités dans la deuxième communication et dont nous aurons besoin chez la démonstration du lemme 1 dans § 4 et dans la quatrième communication.

§ 2. *Démonstration du lemme 5.* Nous démontrons d'abord l'identité:

$$\left.\begin{aligned}&\frac{1}{r!}\int_0^{D_{n-r-\sigma}} du_{r+1+\sigma}\int_0^{D_{n-r-\sigma+1}} du_{r+\sigma}\cdots\int_0^{D_{n-r}}\left(1-A\sum_{\nu=r+1}^{n+1}u_\nu\right)^r du_{r+1}=\\&=\frac{\left(1-A\sum_{\nu=r+2+\sigma}^{n+1}u_\nu\right)^{r+1+\sigma}}{A^{\sigma+1}(r+1+\sigma)!}+\sum_{\mu=0}^{\sigma}\frac{(-1)^{\sigma+1-\mu}\left(\sum_{\nu=r+2+\sigma}^{n+1}u_\nu\right)^{\sigma-\mu}}{(\sigma-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}\end{aligned}\right\}\quad(1)$$

$(0\leqq\sigma\leqq n-r)$; ici 0^e désigne 1.

Cette égalité est valable pour $\sigma=0$, car on a

$$\frac{1}{r!}\int_0^{D_{n-r}}\left(1-A\sum_{\nu=r+1}^{n+1}u_\nu\right)^r du_{r+1}=-\frac{\left(1-A\sum_{\nu=r+1}^{n+1}u_\nu\right)^{r+1}}{(r+1)!\,A}\Bigg|_{u_{r+1}=0}^{u_{r+1}=B-\sum_{\nu=r+2}^{n+1}u_\nu}$$

$$=\frac{\left(1-A\sum_{\nu=r+2}^{n+1}u_\nu\right)^{r+1}}{A(r+1)!}-\frac{(1-AB)^{r+1}}{A(r+1)!}.$$

Supposons que l'identité (1) ait été déjà démontrée pour une valeur de σ avec $0\leqq\sigma\leqq n-r-1$. Alors nous la démontrerons aussi, si σ est remplacé par $\sigma+1$.

De (1) on déduit

$$\frac{1}{r!}\int_0^{D_{n-r-\sigma-1}} du_{r+2+\sigma}\int_0^{D_{n-r-\sigma}} du_{r+1+\sigma}\cdots\int_0^{D_{n-r}}\left(1-A\sum_{\nu=r+1}^{n+1}u_\nu\right)^r du_{r+1}=$$

$$=-\frac{\left(1-A\sum_{\nu=r+2+\sigma}^{n+1}u_\nu\right)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}\Bigg|_{u_{r+2+\sigma}=0}^{u_{r+2+\sigma}=B-\sum_{\nu=r+3+\sigma}^{n+1}u_\nu}+$$

$$+\sum_{\mu=0}^{\sigma}\frac{(-1)^{\sigma+1-\mu}\left(\sum_{\nu=r+2+\sigma}^{n+1}u_\nu\right)^{\sigma+1-\mu}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}\Bigg/_{u_{r+2+\sigma}=0}^{u_{r+2+\sigma}=B-\sum_{\nu=r+3+\sigma}^{n+1}u_\nu}$$

$$=\frac{\left(1-A\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}-\frac{(1-AB)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}+$$

$$+\sum_{\mu=0}^{\sigma}\frac{(-1)^{\sigma+1-\mu}\left\{B^{\sigma+1-\mu}-\left(\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{\sigma+1-\mu}\right\}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}$$

$$=\frac{\left(1-A\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}-\frac{(1-AB)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}+$$

$$+\sum_{\mu=0}^{\sigma+1}\frac{(-1)^{\sigma+2-\mu}\left(\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{\sigma+1-\mu}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}$$

$$+\sum_{\mu=0}^{\sigma+1}\frac{(-1)^{\sigma+1-\mu}B^{\sigma+1-\mu}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}.$$

En changeant l'ordre de sommation, en posant après cela $\mu=h+k$ et en changeant l'ordre de sommation encore, on trouve que la somme

$$\sum_{\mu=0}^{\sigma+1}\frac{(-1)^{\sigma+1-\mu}B^{\sigma+1-\mu}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}$$

est égale à

$$\sum_{h=0}^{\sigma+1}\frac{B^{\sigma+1-h}(1-AB)^{h+r+1}}{(h+r+1)!\,A^{h+1}}\sum_{k=0}^{\sigma+1-h}\frac{(-1)^{\sigma+1-h-k}}{k!\,(\sigma+1-h-k)!}=\frac{(1-AB)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}.$$

Ainsi on obtient

$$\frac{1}{r!}\int_0^{D_{n-r-\sigma-1}}du_{r+2+\sigma}\int_0^{D_{n-r-\sigma}}du_{r+1+\sigma}\ldots\int_0^{D_{n-r}}\left(1-A\sum_{\nu=r+1}^{n+1}u_\nu\right)^r du_{r+1}=$$

$$=\frac{\left(1-A\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{r+2+\sigma}}{A^{\sigma+2}(r+2+\sigma)!}+\sum_{\mu=0}^{\sigma+1}\frac{(-1)^{\sigma+2-\mu}\left(\sum_{\nu=r+3+\sigma}^{n+1}u_\nu\right)^{\sigma+1-\mu}}{(\sigma+1-\mu)!}\sum_{k=0}^{\mu}\frac{B^k(1-AB)^{\mu+r+1-k}}{A^{\mu+1-k}(\mu+r+1-k)!\,k!}$$

et c'est (1) avec $\sigma+1$ au lieu de σ; donc l'identité (1) est démontrée complètement.

En appliquant (1) avec $\sigma = n-r$, nous obtenons

$$J = \frac{1}{A^{n+1-r}(n+1)!} - \sum_{k=0}^{n-r} \frac{B^k (1-AB)^{n+1-k}}{A^{n+1-r-k}(n+1-k)!\,k!}$$

$$= \frac{1}{A^{n+1-r}(n+1)!} - \frac{1}{(n+1)!} \sum_{k=0}^{n-r} \binom{n+1}{k} \frac{B^k (1-AB)^{n+1-k}}{A^{n+1-r-k}},$$

et le lemme est démontré.

§ 3. *Démonstration du lemme 6.* Si un point $(u_1, \ldots, u_{n+1})$ appartient à l'ensemble S, pour chaque des 2^{n+1} combinations des signes $\pm$ le point $(\pm u_1, \ldots, \pm u_{n+1})$ fait partie de S aussi. Par conséquent l'ensemble S se compose de 2^{n+1} cellules, dont chaque cellule est congruente à celle, où $u_1 \geqq 0, \ldots, u_{n+1} \geqq 0$. Si nous désignons cette cellule par C, et son volume par V_C, on obtient

$$V = 2^{n+1} V_C. \quad \ldots\ldots\ldots\ldots \quad (2)$$

On a

$$\left.\begin{aligned} V_C = J_1 + J_2, \text{ avec } J_1 &= \int\int_{G_1} \ldots \int du_1 \ldots du_{n+1} \text{ et} \\ J_2 &= \int\int_{G_2} \ldots \int du_1 \ldots du_{n+1}, \end{aligned}\right\} \quad \ldots \quad (3)$$

où G_1 désigne la partie de la cellule C avec

$$\sum_{\nu=r+1}^{n+1} u_\nu t \leqq \left(\frac{n+1}{2r}\right)^{\frac{r}{n+1-r}}$$

et G_2 désigne la partie avec

$$\left(\frac{n+1}{2r}\right)^{\frac{r}{n+1-r}} < \sum_{\nu=r+1}^{n+1} u_\nu t < 1.$$

Si l'on pose

$$D_\sigma = \frac{1}{t}\left(\frac{n+1}{2r}\right)^{\frac{r}{n+1-r}} - \sum_{\nu=n+2-\sigma}^{n+1} u_\nu \quad (0 \leqq \sigma \leqq n-r)$$

et

$$E_\mu = a\left(1 - \frac{2^{\frac{n+1}{n+1-r}} t^{\frac{r}{n+1-r}} (n+1-r) \sum_{\nu=r+1}^{n+1} u_\nu t}{(n+1)^{\frac{n+1}{n+1-r}}}\right) - \sum_{\nu=r+1-\mu}^{r} u_\nu \quad (0 \leqq \mu \leqq r-1)$$

(une somme vide soit égale à zéro),

on a

$$J_1 = \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} du_{r+1} \int_0^{E_0} du_r \int_0^{E_1} du_{r-1} \dots \int_0^{E_{r-1}} du_1$$

$$= \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} du_{r+1} \int_0^{E_0} du_r \int_0^{E_1} du_{r-1} \dots \int_0^{E_{r-2}} E_{r-1}\, du_2.$$

En vertu de

$$\int_0^{E_{r-\mu-1}} E_{r-\mu}^{\mu}\, du_{\mu+1} = \int_0^{E_{r-\mu-1}} (E_{r-\mu-1} - u_{\mu+1})^{\mu}\, du_{\mu+1} =$$

$$= -\frac{1}{\mu+1} (E_{r-\mu-1} - u_{\mu+1})^{\mu+1} \Big|_0^{E_{r-\mu-1}} = \frac{1}{\mu+1} E_{r-\mu-1}^{\mu+1} \qquad (1 \leqq \mu \leqq r-1),$$

on trouve

$$J_1 = \frac{1}{2!} \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} du_{r+1} \int_0^{E_0} du_r \int_0^{E_1} du_{r-1} \dots \int_0^{E_{r-3}} E_{r-2}^2\, du_3$$

$$= \frac{1}{(r-1)!} \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} du_{r+1} \int_0^{E_0} E_1^{r-1}\, du_r$$

$$= \frac{a^r}{r!} \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} \left\{ 1 - \frac{2^{\frac{n+1}{n+1-r}}\, t^{\frac{r}{n+1-r}}\, (n+1-r) \sum_{\nu=r+1}^{n+1} u_\nu\, t}{(n+1)^{\frac{n+1}{n+1-r}}} \right\}^r du_{r+1}.$$

Appliquons le lemme 5 avec

$$A = \frac{2^{\frac{n+1}{n+1-r}}\, t^{\frac{r}{n+1-r}}\, (n+1-r)\, t}{(n+1)^{\frac{n+1}{n+1-r}}} \quad \text{et} \quad B = \frac{1}{t} \left(\frac{n+1}{2r} \right)^{\frac{r}{n+1-r}};$$

nous trouvons:

$$J_1 = \frac{a^r}{(n+1)!} \left\{ \frac{(n+1)^{n+1}}{2^{n+1}\, r^r\, (n+1-r)^{n+1-r}\, t^{n+1-r}} \right.$$

$$\left. - \sum_{k=0}^{n-r} \binom{n+1}{k} \frac{\left(\frac{n+1}{2r} \right)^{\frac{rk}{n+1-r}} \left(\frac{2r-n-1}{n+1} \right)^{n+1-k} (n+1)^{\frac{(n+1)(n+1-r-k)}{n+1-r}}}{t^k \left\{ 2^{\frac{n+1}{n+1-r}}\, r^{\frac{r}{n+1-r}}\, (n+1-r)\, t \right\}^{n+1-r-k}} \right\}$$

$$= \frac{a^r}{t^{n+1-r}\, r^r\, (n+1)!} \left\{ \frac{(n+1)^{n+1}}{2^{n+1}\, (n+1-r)^{n+1-r}} - \sum_{k=0}^{n-r} \binom{n+1}{k} \frac{\left(r - \frac{n+1}{2} \right)^{n+1-k}}{(n+1-r)^{n+1-r-k}} \right\}.$$

La somme entre les accolades est égale à (on pose $\mu = n+1-k$)

$$\sum_{\nu=r+1}^{n+1}\binom{n+1}{\mu}\frac{\left(r-\frac{n+1}{2}\right)^{\mu}}{(n+1-r)^{\mu-r}} = \sum_{\mu=0}^{n+1}\binom{n+1}{\mu}\frac{\left(r-\frac{n+1}{2}\right)^{\mu}}{(n+1-r)^{\mu-r}} - \sum_{\mu=0}^{r}\binom{n+1}{\mu}\frac{\left(r-\frac{n+1}{2}\right)^{\mu}}{(n+1-r)^{\mu-r}}$$

$$= (n+1-r)^{r}\left(1+\frac{r-\frac{n+1}{2}}{n+1-r}\right)^{n+1} - \sum_{\mu=0}^{r}\binom{n+1}{\mu}\left(r-\frac{n+1}{2}\right)^{\mu}(n+1-r)^{r-\mu};$$

$$= \frac{(n+1)^{n+1}}{2^{n+1}(n+1-r)^{n+1-r}} - \sum_{\mu=0}^{r}\binom{n+1}{\mu}\left(r-\frac{n+1}{2}\right)^{\mu}(n+1-r)^{r-\mu};$$

on a donc

$$J_1 = \frac{a^r}{t^{n+1-r}\,r^r\,(n+1)!}\sum_{\mu=0}^{r}\binom{n+1}{\mu}\left(r-\frac{n+1}{2}\right)^{\mu}(n+1-r)^{r-\mu}. \quad (4)$$

Si l'on pose:

$$C = \left(\frac{n+1}{2r}\right)^{\frac{r}{n+1-r}} \quad \text{et} \quad F_k = a\left\{\left(\frac{1}{t\sum_{\nu=r+1}^{n+1}u_\nu}\right)^{\frac{n+1-r}{r}} - 1\right\} - \sum_{\nu=r+1-k}^{r}u_\nu \quad (0 \leqq k \leqq r-1),$$

on trouve pour J_2:

$$J_2 = \sum_{\mu=0}^{n-r}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \cdots \int_0^{\frac{1}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu} du_{r+1+\mu}\int_0^{\frac{1}{t}-\sum_{\nu=r+1+\mu}^{n+1}u_\nu} du_{r+\mu}\cdots\int_0^{\frac{1}{t}-\sum_{\nu=r+2}^{n+1}u_\nu} du_{r+1}\int_0^{F_0} du_r\int_0^{F_1} du_{r-1}\cdots\int_0^{F_{r-1}} du_1.$$

En vertu de

$$\int_0^{F_{r-k-1}} F_{r-k}^{k}\,du_{k+1} = \int_0^{F_{r-k-1}} (F_{r-k-1}-u_{k+1})^k\,du_{k+1} =$$

$$= -\frac{1}{k+1}(F_{r-k-1}-u_{k+1})^{k+1}\Big|_0^{F_{r-k-1}} = \frac{F_{r-k-1}^{k+1}}{k+1} \quad (1 \leqq k \leqq r-1),$$

on trouve que l'intégrale

$$\int_0^{F_0} du_r\int_0^{F_1} du_{r-1}\cdots\int_0^{F_{r-1}} du_1$$

est égale à

$$\int_0^{F_0} du_r \int_0^{F_1} du_{r-1} \dots \int_0^{F_{r-2}} F_{r-1}\, du_2 = \frac{a^r}{r!}\left\{\left(\frac{1}{t\sum_{\nu=r+1}^{n+1} u_\nu}\right)^{\frac{n+1-r}{r}} - 1\right\}^r,$$

et donc

$$J_2 = \frac{a^r}{r!}\sum_{\mu=0}^{n-r}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \dots \int_{\frac{C}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu}^{\frac{1}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu} du_{r+1+\mu}\int_0^{\frac{1}{t}-\sum_{\nu=r+1+\mu}^{n+1}u_\nu} du_{r+\mu}\dots\int_0^{\frac{1}{t}-\sum_{\nu=r+2}^{n+1}u_\nu}\left\{\left(\frac{1}{t\sum_{\nu=r+1}^{n+1}u_\nu}\right)^{\frac{n+1-r}{r}}-1\right\}^r du_{r+1}$$

Si l'on substitue successivement

$$\left(\frac{1}{t\sum_{\nu=r+\lambda}^{n+1}u_\nu}\right)^{\frac{n+1-r}{r}} - 1 = v_\lambda \qquad (1 \leqq \lambda \leqq \mu+1),$$

on peut écrire pour l'intégrale J_2:

$$J_2 = \frac{a^r r}{r!\,t(n+1-r)}\sum_{\mu=0}^{n-r}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \dots \int_{\frac{C}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu}^{\frac{1}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu} du_{r+1+\mu}\int_0^{\frac{1}{t}-\sum_{\nu=r+1+\mu}^{n+1}u_\nu} du_{r+\mu}\dots\int_0^{\frac{1}{t}-\sum_{\nu=r+3}^{n+1}u_\nu} du_{r+2}\left\{\int_0^{v_2}\frac{v_1^r\,dv_1}{(v_1+1)^{\frac{n+1}{n+1-r}}}\right\}$$

$$= \frac{a^r r^2}{r!\,t^2(n+1-r)^2}\sum_{\mu=0}^{n-r}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \dots \int_{\frac{C}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu}^{\frac{1}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu} du_{r+1+\mu}\int_0^{\frac{1}{t}-\sum_{\nu=r+1+\mu}^{n+1}u_\nu} du_{r+\mu}\dots\int_0^{\frac{1}{t}-\sum_{\nu=r+4}^{n+1}u_\nu} du_{r+3}$$

$$\left\{\int_0^{v_3}\frac{dv_2}{(v_2+1)^{\frac{n+1}{n+1-r}}}\int_0^{v_2}\frac{v_1^r\,dv_1}{(v_1+1)^{\frac{n+1}{n+1-r}}}\right\}$$

$$= \frac{a^r}{r!}\sum_{\mu=0}^{n-r}\frac{r^\mu}{t^\mu(n+1-r)^\mu}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \dots \int_{\frac{C}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu}^{\frac{1}{t}-\sum_{\nu=r+2+\mu}^{n+1}u_\nu} du_{r+1+\mu}\left\{\int_0^{v_{\mu+1}}\frac{dv_\mu}{(v_\mu+1)^{\frac{n+1}{n+1-r}}}\dots\int_0^{v_2}\frac{v_1^r\,dv_1}{(v_1+1)^{\frac{n+1}{n+1-r}}}\right\}$$

$$= \frac{a^r}{r!}\sum_{\mu=0}^{n-r}\frac{r^{\mu+1}P_\mu}{t^{\mu+1}(n+1-r)^{\mu+1}}\int_0^{\frac{C}{t}} du_{n+1}\int_0^{\frac{C}{t}-u_{n+1}} du_n \dots \int_0^{\frac{C}{t}-\sum_{\nu=r+3+\mu}^{n+1}u_\nu} du_{r+2+\mu}$$

$$= \frac{a^r}{r!} \sum_{\mu=0}^{n-r} \frac{t^{\mu+1} P_\mu}{t^{\mu+1}(n+1-r)^{\mu+1}} \int_0^{\frac{C}{t}} du_{n+1} \int_0^{\frac{C}{t}-u_{n+1}} du_n \ldots \int_0^{\frac{C}{t}-\sum_{\nu=r+4+\mu}^{n+1} u_\nu} \left(\frac{C}{t} - \sum_{\nu=r+3+\mu}^{n+1} u_\nu\right) du_{r+3+\mu}$$

$$= \frac{a^r}{r!} \sum_{\mu=0}^{n-r} \frac{t^{\mu+1} P_\mu C^{n-r-\mu}}{t^{\mu+1}(n+1-r)^{\mu+1}(n-r-\mu)!\, t^{n-r-\mu}}.$$

Donc on a

$$J_2 = \frac{a^r}{t^{n+1-r} r!} \sum_{\mu=0}^{n-r} \frac{r^{\mu+1}}{(n+1-r)^{\mu+1}(n-r-\mu)!} \left(\frac{n+1}{2r}\right)^{\frac{r(n-r-\mu)}{n+1-r}} P_\mu. \quad . \quad (5)$$

D'après (2), (3), (4) et (5) il suit

$$V = 2^{n+1} V_C = 2^{n+1}(J_1 + J_2) = \frac{a^r \varrho_{n,r}}{t^{n+1-r}}$$

et le lemme est démontré.

§ 4. *Démonstration du lemme 1. (Première partie.)* Nous réprésentons l'espace R_{n+1} des points $(x_1, \ldots, x_{n+1})$ sur l'espace R'_{n+1} des points $(u_1, \ldots, u_{n+1})$ par la transformation:

$$u_\nu = L_\nu \qquad (\nu = 1, \ldots, n+1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

En substituant en (6) à $(x_1, \ldots, x_{n+1})$ tous les points à coordinées entières de R_{n+1}, on obtient un système des points $(u_1, \ldots, u_{n+1})$, que nous voulons considérer comme les points spéciaux de R'_{n+1} dans le sens du lemme 4. Parce que ces points forment un système dénombrable, on peut choisir dans R'_{n+1} un point $(a_1, \ldots, a_{n+1})$, qui diffère en toutes ses coordinées avec tous les points spéciaux. Considérons maintenant dans R'_{n+1} les parallélépipèdes R, bornés par les „plans":

$$u_\nu = a_\nu + k_\nu g_\nu b_{\nu\nu} \qquad (\nu = 1, \ldots, n+1),$$

où $k_1, \ldots, k_{n+1}$ sont des nombres naturels donnés, et $g_1, \ldots, g_{n+1}$ des nombres parcourant indépendant l'un de l'autre toutes les valeurs entières. Il est clair, que chaque R a le volume

$$W = k_1 \ldots k_{n+1} \prod_{\nu=1}^{n} b_{\nu\nu} = k_1 \ldots k_{n+1} \triangle, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

et qu'il contient

$$k = k_1 \ldots k_{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

points spéciaux de R'_{n+1}, puisqu'il y a, les nombres $g_1, \ldots, g_{n+1}$ étant supposés fixes, précisément $k_1 \ldots k_{n+1}$ systèmes de nombres entiers $(x_1, \ldots, x_{n+1})$ qui vérifient les inégalités:

$$\begin{aligned}
a_1 + k_1 g_1 b_{11} < b_{11} x_1 &\qquad < a_1 + k_1 (g_1 + 1) b_{11} \\
a_2 + k_2 g_2 b_{22} < b_{21} x_1 + b_{22} x_2 &\qquad < a_2 + k_2 (g_2 + 1) b_{22} \\
\ldots \quad . \quq
\end{align}$$

A. *Supposons d'abord* $r \geqq \frac{n+1}{2}$.

Dans l'espace R'_{n+1} nous désignons l'ensemble S' par les inégalités:

$$\sum_{\nu=r+1}^{n+1} |u_\nu| t < 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

$$\sum_{\nu=1}^{r} \frac{|u_\nu|}{a} + \frac{2^{\frac{n+1}{n+1-r}} t^{\frac{r}{n+1-r}} (n+1-r) \sum_{\nu=r+1}^{n+1} |u_\nu| t}{(n+1)^{\frac{n+1}{n+1-r}}} < 1, \text{ si } \sum_{\nu=r+1}^{n+1} |u_\nu| t \leqq \left(\frac{n+1}{2r}\right)^{\frac{r}{n+1-r}}, \quad (10)$$

$$\sum_{\nu=r+1}^{n+1} |u_\nu| t \left\{ \sum_{\nu=1}^{r} \frac{|u_\nu|}{a} + 1 \right\}^{\frac{r}{n+1-r}} < 1, \text{ si } \left(\frac{n+1}{2r}\right)^{\frac{r}{n+r-1}} < \sum_{\nu=r+1}^{n+1} |u_\nu| t < 1, \quad (11)$$

où nous avons posé

$$a = \sqrt[r]{\frac{t^{n+1-r} \Delta}{\varrho_{n,r}}}. \quad \ldots\ldots\ldots\ldots \quad (12)$$

En vertu du lemme 6 le volume V de S' est égal à $V = \frac{a^r \varrho_{n,r}}{t^{n+1-r}}$; en tenant compte de (12) on trouve

$$V = \Delta. \quad \ldots\ldots\ldots\ldots \quad (13)$$

Selon le lemme 4 à tout nombre positif ε dans l'espace R'_{n+1} une translation:

$$z_\nu = u_\nu + d_\nu \quad (\nu = 1, \ldots, n+1), \quad \ldots\ldots \quad (14)$$

correspond, par laquelle l'ensemble S' est transféré dans une telle position, que le nombre des points spéciaux de R'_{n+1}, qui se trouvent à l'intérieur de S' ou à l'intérieur d'une sphère de rayon ε et avec un point-frontière de S' comme centre, est supérieur à $\frac{Vk}{W}$, et donc supérieur à $\frac{\Delta k_1 \ldots k_{n+1}}{k_1 \ldots k_{n+1} \Delta} = 1$ en vertu de (7), (8) et (13), de sorte qu'on a au moins deux de tels points spéciaux:

$$(z'_1, \ldots, z'_{n+1}) \text{ et } (z''_1, \ldots, z''_{n+1}). \quad \ldots\ldots\ldots \quad (15)$$

Désignons par $(u'_1, \ldots, u'_{n+1})$ et $(u''_1, \ldots, u''_{n+1})$ les points, qui correspondent aux points (15) d'après la translation (14), et par $(x'_1, \ldots, x'_{n+1})$ et $(x''_1, \ldots, x''_{n+1})$ les points à coordinées entières de R_{n+1}, qui correspondent aux points (15) d'après (6). Finalement nous désignons par $L'_1, \ldots, L'_{n+1}$ et $L''_1, \ldots, L''_{n+1}$ les formes linéaires, indiquées dans le lemme 1, correspondant aux points $(x'_1, \ldots, x'_{n+1})$ et $(x''_1, \ldots, x''_{n+1})$.

(*À suivre*).

Mathematics. — *Die Begründung der Trigonometrie in der hyperbolischen Ebene.* (Zweite Mitteilung.) Von J. C. H. GERRETSEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 2. Die hyperbolischen Bewegungen.

Wir haben schon im vorigen Paragraphen bemerkt, dasz eine Bewegung eine umkehrbar eindeutige Endenzuordnung hervorruft. Wir wollen uns jetzt mit der Aufgabe beschäftigen diese Zuordnung formelmäszig zu fassen.

Zunächst werden wir beweisen:

1. · *Zu einer vorgegebenen nicht-singulären gebrochenen linearen Transformation der Enden gibt es eine eindeutig bestimmte Bewegung, welche eben diese Endentransformation hervorruft.*

Es sei also jedem Ende ξ vermöge:

$$\bar{\xi} = \frac{\lambda\xi + \mu}{\lambda'\xi + \mu'}, \quad \begin{vmatrix} \lambda & \mu \\ \lambda' & \mu' \end{vmatrix} \neq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,1)$$

ein Ende $\bar{\xi}$ zugeordnet. Wenn $\lambda' \neq 0$, dann können wir $\lambda' = 1$ annehmen. Wir haben dann:

$$\bar{\xi} = \lambda + \frac{\mu - \lambda\mu'}{\xi + \mu'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,2)$$

Ist aber $\lambda' = 0$, dann ist $\mu' \neq 0$ und wir können $\mu' = 1$ annehmen. In diesem Falle haben wir:

$$\bar{\xi} = \lambda\xi + \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,3)$$

Es ist sofort ersichtlich, dasz man in beiden Fällen die durch diese Formeln zum Ausdruck gebrachte Endenzuordnung erzielen kann, wenn man die Transformationen:

$$\bar{\xi} = -\xi, \ \bar{\xi} = \frac{1}{\xi}, \ \bar{\xi} = \xi + \nu, \ \bar{\xi} = \varkappa\xi, \ (\varkappa > 0), \quad . \quad . \quad . \quad (2,4)$$

in geeigneter Reihenfolge endlich oft nach einander anwendet. Jeder dieser Transformationen entspricht aber eine Bewegung, welche die Transformation eben erzeugt und zwar hintereinander:

$$\mathfrak{S}_0, \mathfrak{P}_1, \mathfrak{S}_{\frac{1}{2}}, \mathfrak{S}_0, \mathfrak{P}_{\sqrt{\varkappa}}, \mathfrak{P}_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,5)$$

deren Zusammensetzung die fragliche Bewegung liefert. Die eindeutige Bestimmtheit der Bewegung, welche eine vorgegebene Transformation (2,1) erzeugt, erfolgt sofort aus der Tatsache, dasz eine Bewegung, welche jedes Ende fest läszt, auch jeden Punkt ungeändert läszt. Denn sollte bei einer solchen Bewegung z.B. der Punkt A in den Punkt B übergeführt werden, dann würde die Senkrechte in A auf der Geraden AB in die Senkrechte in B auf dieser Geraden übergeführt. Diese Senkrechten hätten dann ein Ende gemeinsam, was nicht möglich ist. Damit ist der Beweis des Satzes beendet.

Mit den homographischen Endentransformationen erhält man aber auch alle Bewegungen, denn es gilt:

2. *Durch irgendeine Bewegung erleiden die Enden eine nicht-singuläre gebrochene lineare Transformation. Die dabei auftretenden Koeffizienten sind bis auf einen gemeinsamen Faktor bestimmt.*

Bekanntlich kann jede Bewegung in eine endliche Folge von Spiegelungen zerlegt werden. Wir brauchen uns darum nur um den Fall einer Spiegelung zu kümmern.

Es seien α und α' die Enden der Geraden an der die Spiegelung stattfindet. Es seien η und $\bar{\eta}$ irgend zwei Enden, die durch eine Spiegelung an dieser Geraden einander zugeordnet werden. Die Endentransformation:

$$\bar{\xi} = \frac{\xi - \alpha}{\xi - \alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,6)$$

bestimmt eine Bewegung, bei der dem Ende α das Ende 0 und dem Ende α' das Ende ∞ entspricht. Das Ende η bezw. $\bar{\eta}$ wird in η' bezw. $\bar{\eta}'$ übergeführt. Nun liegen die Enden η' und $\bar{\eta}'$ symmetrisch inbezug auf die Gerade $(0, \infty)$, so dasz $\bar{\eta}' = -\eta'$. Daraus folgt aber:

$$\frac{\bar{\eta} - \alpha}{\bar{\eta} - \alpha'} = -\frac{\eta - \alpha}{\eta - \alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,7)$$

Diese Beziehung kann man umformen zu:

$$\bar{\eta} = \frac{\frac{1}{2}(\alpha + \alpha')\,\eta - \alpha\,\alpha'}{\eta - \frac{1}{2}(\alpha + \alpha')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,8)$$

also eine gebrochene lineare Substitution mit nicht verschwindender Determinante. Durch Zusammensetzung von endlich vielen Transformationen der Gestalt (2,8) entsteht aber wieder eine nicht-singuläre homographische Transformation. Die vorgegebene Bewegung ordnet den Enden 0, 1 und ∞ wohlbestimmte Enden zu. Dadurch sind aber die in (2,1) auftretenden Koeffizienten bis auf einen gemeinsamen Faktor bestimmt, w.z.b.w.

Die beiden folgenden Hilfssätze werden uns noch oft gute Dienste leisten.

3. *Eine Gerade (α, α') geht dann und nur dann durch den Punkt O, wenn die Relation:*

$$\alpha\,\alpha' = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,9)$$

besteht.

Dabei können wir $\alpha \neq 1$ voraussetzen. Eine Gerade (α, α') durch O wird durch die Bewegung $\mathfrak{P}_1\,\mathfrak{S}_0$ nicht geändert, während α in $-\frac{1}{\alpha} = \alpha'$ übergeführt wird. Ist umgekehrt die Gerade (α, α') mit $\alpha' = -\frac{1}{\alpha}$ vorgelegt, dann bleibt diese bei der Bewegung $\mathfrak{P}_1\,\mathfrak{S}_0$ ungeändert, da ja die Enden vertauscht werden, während diese Bewegung keine nicht durch O gehende Gerade ungeändert läszt.

Allgemeiner gilt:

4. *Eine Gerade (α, α') geht dann und nur dann durch den Punkt P auf der Geraden $(0, \infty)$, wenn die Relation:*

$$\alpha\,\alpha' = -\pi^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,10)$$

besteht, wobei π und $-\pi$ die Enden der Senkrechten in P auf der Geraden $(0, \infty)$ sind.

Die Richtigkeit des Satzes wird sofort eingesehen, wenn man mit Hilfe der Bewegung $\mathfrak{P}_{\sqrt{\frac{1}{\pi}}}\,\mathfrak{P}_1$ den Punkt P in den Punkt O überführt und vorigen Satz anwendet. Wir haben dabei π als positiv vorausgesetzt, was natürlich keine Beschränkung bedeutet.

Wir wollen nun eine wichtige Art von Bewegungen betrachten, die Drehungen um den Punkt O.

Mit $\mathfrak{R}_\alpha$ bezeichnen wir die Spiegelung an der Geraden $\left(\alpha, -\frac{1}{\alpha}\right)$, wobei α positiv ist. Die Bewegung $\mathfrak{R}_\alpha\,\mathfrak{R}_1$ wird *Drehung* um den Punkt O genannt. Wir behaupten:

5. *Bei einer Drehung um den Punkt O erleiden die Enden eine Transformation von der Form:*

$$\bar{\xi} = \frac{\xi + \vartheta}{1 - \xi\,\vartheta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,11)$$

Umgekehrt bestimmt eine solche Transformation, wobei ϑ willkürlich vorgegeben ist, eine Drehung um den Punkt O.

Setzen wir in (2,7) $\alpha' = -\frac{1}{\alpha}$, so kommt:

$$\bar{\eta} = \frac{\vartheta\,\eta + 1}{\eta - \vartheta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,12)$$

wenn wir $\vartheta = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)$ setzen. Damit haben wir die von $\mathfrak{R}_\alpha$ herrührende Transformation schon gefunden. Wenn wir noch $\bar{\eta} = \bar{\xi}$ und $\eta = \frac{1}{\xi}$ setzen, erhalten wir:

$$\bar{\xi} = \frac{\xi + \vartheta}{1 - \xi\,\vartheta}$$

also die von $\mathfrak{R}_\alpha\,\mathfrak{R}_1$ erzeugte Transformation. Die Abänderung, welche die Formel erfährt für $\alpha = \infty$, brauchen wir wohl nicht besonders hervorzuheben.

Es sei nun umgekehrt eine Transformation (2,11) vorgelegt. Diese Transformation setzt sich zusammen aus der von $\mathfrak{R}_1$ erzeugten Transformation $\eta = \frac{1}{\xi}$ und der Transformation (2,12). Für $\vartheta = \infty$ haben wir $\bar{\eta} = -\eta$, also eine Spiegelung, die wir mit $\mathfrak{R}_\infty$ bezeichnen können. Ist aber $\vartheta \neq \infty$, dann kann nur (2,12) von der Spiegelung $\mathfrak{R}_\alpha$ erzeugt werden, wenn wir α aus der Gleichung:

$$\frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) = \vartheta,$$

oder

$$\alpha^2 - 2\,\alpha\,\vartheta - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2,13)$$

bestimmen können. Diese Gleichung ist aber in $\mathfrak{h}$ lösbar, da ihre Diskriminante stets positiv ist. Auszerdem hat das Produkt der Wurzeln α und α' den Wert -1, so dasz die Gerade (α, α') tatsächlich durch O geht. Damit ist der Beweis aber fertig.

Wir können die Drehung um O positiv oder negativ nennen, je nachdem ϑ positiv bezw. negativ ist. Für $\vartheta = 1$ erhalten wir eine positive Drehung um O mit einem rechten Winkel; für $\vartheta = -1$ erhalten wir eine negative Drehung um O mit einem rechten Winkel.

§ 3. Die hyperbolischen Funktionen.

Da die Beziehung „kongruent" reflexiv, symmetrisch und transitiv ist, können wir im Bereiche aller Strecken disjunkte Klassen von untereinander kongruenten Strecken bilden. Wird die Streckenklasse a_1 von der Strecke A_1B_1 erzeugt und die Streckenklasse a_2 von der Strecke A_2B_2, so heiszt die Klasse a_3, welche von einer Strecke A_3B_3 erzeugt wird, die kongruent ist mit der Summe der Strecken A_1B_1 und A_2B_2, die Summe $a_1 + a_2$ der Streckenklassen a_1 und a_2. Zwischen den Streckenklassen kann eine Ordnungsbeziehung definiert werden und zwar heiszt die Klasse a_1 gröszer als die Klasse a_2, $a_1 > a_2$, wenn a_1 von einer Strecke erzeugt wird, die gröszer ist als eine die Klasse a_2 erzeugende Strecke. Zu den Klassen a_1 und a_2, mit $a_1 > a_2$, existiert immer eine Streckenklasse $x = a_1 - a_2$, wofür gilt: $a_1 = a_2 + x$. Die Klasse x heiszt die Differenz der Klassen a_1 und a_2. Diese Begriffsbildungen sind invariant gegenüber Bewegungen.

Es sei nun irgend eine Streckenklasse a vorgelegt und es sei AB eine die Klasse erzeugende Strecke. Wir errichten auf der Geraden AB die Senkrechten in A und in B. Die Enden der ersten Senkrechten seien α und α' und die Enden der zweiten Senkrechten seien β und β'. Dabei werde vorausgesetzt, dasz die Enden α und β auf derselben Seite der Geraden AB liegen. Wir bilden nun das Doppelverhältnis:

$$\delta = \begin{bmatrix} \alpha\, \alpha' \\ \beta\, \beta' \end{bmatrix} = \frac{\alpha - \beta}{\alpha' - \beta} \cdot \frac{\alpha' - \beta'}{\alpha - \beta'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,1)$$

Wir lassen zu, dasz eines der auftretenden Enden ∞ ist, insofern die üblichen Verabredungen über derartige Doppelverhältnisse getroffen werden. Offenbar hat δ für jede Strecke der Klasse a denselben Wert, vermöge der Invarianz eines Doppelverhältnisses gegenüber gebrochenen linearen Transformationen. Jeder Klasse a kann also ein δ zugeordnet werden.

Wir wollen nun eine Strecke OA der Klasse a betrachten, wobei A auf der von O nach ∞ gehenden Halbgeraden liegt. Wenn α und $-\alpha$ die Enden der Senkrechten in A auf der Geraden $(0, \infty)$ sind, so ist:

$$\delta = \begin{bmatrix} \alpha & -\alpha \\ 1 & -1 \end{bmatrix} = \left(\frac{\alpha - 1}{\alpha + 1}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,2)$$

und daraus geht hervor, dasz δ stets positiv ist. Wenn wir nun α als positiv voraussetzen, dann ist:

$$\alpha = \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,3)$$

Dem von der Streckenklasse a eindeutig bestimmten Ausdruck im rechten Gliede wollen wir unsere besondere Aufmerksamkeit widmen und mit $\exp a$ bezeichnen. Offenbar gilt:

1. *Für jede Streckenklasse a ist* $\exp a$ *bestimmt und es ist* $\exp a > 1$.

Und umgekehrt:

2. *Wenn irgend ein Ende $\alpha > 1$ gegeben ist, dann gibt es immer eine eindeutig bestimmte Streckenklasse a mit* $\exp a = \alpha$.

Denn die Klasse a wird erzeugt von der Strecke OA, wobei A der Fuszpunkt des Lotes aus α auf der Geraden $(0, \infty)$ ist.

Wir werden nun zeigen, dasz die Funktion exp formell mit der Exponentialfunktion der reellen Analysis übereinstimmt. Wir beweisen zunächst:

3. *Für die Funktion* exp *besteht das Additionstheorem*:

$$\exp(a + b) = \exp a \,.\, \exp b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,4)$$

Es sei OA eine Strecke aus der Klasse a und OB eine Strecke aus der Klasse b. Dabei werde vorausgesetzt, dasz die Punkte A und B auf der von O nach ∞ gehenden Halbgeraden liegen. Die Enden der Senkrechten in A auf der Geraden $(0, \infty)$ seien α und $-\alpha$ und die Enden der Senkrechten in B seien β und $-\beta$; dabei werden α und β als positiv vorausgesetzt.

Die Bewegung $\mathfrak{B}_{\sqrt{\beta}}\,\mathfrak{B}_1$ ordnet dem Punkt A den Punkt C zu und es gilt die Streckenrelation:

$$OC = OA + OB. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,5)$$

Die Strecke OC erzeugt die Klasse $a + b$. Es seien nun γ und $-\gamma$ die Enden der Senkrechten im Punkte C auf der Geraden $(0, \infty)$, wobei γ positiv ist. Die genannte Bewegung erzeugt die Endentransformation:

$$\bar{\xi} = \xi\, \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,6)$$

und führt also das Ende α in das Ende

$$\gamma = \alpha\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,7)$$

über. Da aber, wie oben gezeigt wurde, $\gamma = \exp(a+b)$, $\alpha = \exp a$ und $\beta = \exp b$ ist, haben wir damit die Formel (3,4) bewiesen.

Es gilt weiter:

4. *Ist die Streckenklasse a gröszer als die Streckenklasse b, dann gilt:*

$$\exp(a-b) = \frac{\exp a}{\exp b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,8)$$

Der Beweis wird fast ebenso geführt wie beim vorigen Satz. Wir müssen nun aber die Bewegung $\mathfrak{P}_{\sqrt{\frac{1}{\beta}}}\,\mathfrak{P}_1$ anwenden.

Wir können uns von der Einschränkung $a > b$ befreien, wenn wir beachten:

5. *Die Menge der Streckenklassen läszt sich zu einer additiven angeordneten Gruppe erweitern.*

Wir können in bekannter Weise vorgehen, indem wir formelle Differenzen $a - b$ für jedes Paar a, b einführen und dafür in üblicher Weise die Rechnungsregeln definieren. Wir werden die so erhaltene Gruppe, welche die Menge der Streckenklassen umfaszt, mit $\mathfrak{S}$ bezeichnen.

Es ist nun leicht ersichtlich, dasz Satz 3 auch für die Elemente der Gruppe $\mathfrak{S}$ gültig ist, wenn wir

$$\exp 0 \,(= \exp(a-a)) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,9)$$

und

$$\exp(b-a) = \frac{1}{\exp(a-b)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,10)$$

setzen. Klar ist dann:

6. *Für jedes Element x der Gruppe* $\mathfrak{S}$ *ist* $\exp x > 0$.

Wir sind jetzt imstande für die Elemente der Gruppe $\mathfrak{S}$ die hyperbolischen Funktionen zu definieren. Wir setzen:

$$\cosh x = \tfrac{1}{2}(\exp x + \exp -x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,11)$$

$$\sinh x = \tfrac{1}{2}(\exp x - \exp -x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,12)$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,13)$$

Wir sehen daraus, dasz für irgend ein Element der Gruppe $\mathfrak{S}$ die hyperbolischen Funktionen Sinn haben und es ist leicht den Wertverlauf dieser Funktionen zu überblicken.

7. *Es bestehen die Additionstheoreme:*

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \quad . \quad . \quad . \quad (3,14)$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y, \quad . \quad . \quad . \quad (3,15)$$

für irgend zwei Elemente der Gruppe $\mathfrak{S}$.

Man beweist diese Formeln mit Hilfe des Additionstheorems der Funktion exp.

Es ist nun ganz leicht die bekannten elementaren Formeln der hyperbolischen Funktionen herzuleiten. Für später bemerken wir noch:

$$\tanh \tfrac{1}{2} a = \frac{\sinh a}{\cosh a + 1} = \sqrt{\delta}, \; a > 0, \quad . \quad . \quad . \quad . \quad (3,16)$$

wenn δ das zu a gehörige Doppelverhältnis ist.

(To be continued)

Mathematics. — *Over reeksen en bepaalde integralen, waarbij functies van* BESSEL *optreden.* II. Door J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of April 25, 1942.)

3. Voor $\mu = \nu - m$, waarin m geheel $\geqq 0$ is, gaat I over in een eindige reeks; vervangen we tevens ν door ϱ, dan krijgen we:

$$\sum_{n=0}^{m} \frac{(-1)^n \binom{m}{n}}{\Gamma(\varrho+n-m+1)} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \frac{\left(\frac{x}{2}\right)^{\varrho+m}}{\Gamma(\varrho+1)} I_{\varrho-m}(x), \quad . \quad (V)$$

ϱ willekeurig, m geheel $\geqq 0$.

Door in (1) tot en met (8) ϱ te vervangen door $\varrho + n$ en daarna beide leden te vermenigvuldigen met $\frac{(-)^n \binom{m}{n}}{2^{\varrho+n}\,\Gamma(\varrho+n-m+1)}$, volgen na sommatie over n van 0 tot m, onder toepassing van V in het linkerlid onder het integraalteeken, voor $R(\nu) > -\frac{1}{2}$, $R(\varrho) > -\frac{1}{2}$ en $m \geqq 0$:

$$\left.\begin{aligned}&\int_0^x I_\nu(x-\alpha)\, I_{\varrho-m}(\alpha)\,(x-\alpha)^\nu\, \alpha^{\varrho+m}\, d\alpha = \\ &= \frac{\Gamma(\nu+\tfrac{1}{2})\, m!\, 2^{\nu+\varrho+m}}{\sqrt{\pi}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\Gamma(\varrho+n+\tfrac{1}{2})}{\Gamma(\nu+\varrho+n+1)} \left(\frac{x}{2}\right)^{\nu+\varrho+n+\frac{1}{2}} I_{\nu+\varrho+n+\frac{1}{2}}(x),\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned}&\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\, I_{\varrho-m}(\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{\varrho+m}\, d\alpha = \\ &= \frac{\Gamma(\nu+1)\, m!\, 2^{\nu+\varrho+m+\frac{1}{2}}}{\sqrt{\pi}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\Gamma(\varrho+n+\tfrac{1}{2})}{\Gamma(\nu+\varrho+n+\tfrac{3}{2})} \left(\frac{x}{2}\right)^{\nu+\varrho+n+1} I_{\nu+\varrho+n}(x),\end{aligned}\right\} \quad (17)$$

$$\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \sin(x-\alpha)\, d\alpha = m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} I_{\varrho+n+1}(x), \quad (18)$$

$$\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \cos(x-\alpha)\, d\alpha = m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} I_{\varrho+n}(x), \quad . \quad (19)$$

$$\left.\begin{aligned}&\int_0^x I_{\varrho-m}(\alpha)\, \alpha^{\varrho+m} \sin\alpha\, d\alpha = \\ &= m!\, 2^{\varrho+m+1} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1} \{\sin x\, I_{\varrho+n}(x) - \cos x\, I_{\varrho+n+1}(x)\},\end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned}&\int_0^x I_{\varrho-m}(\alpha)\,\alpha^{\varrho+m}\cos\alpha\,d\alpha = \\ &= m!\,2^{\varrho+m+1}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho}{m-n}}{n!}\cdot\frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1}\{\cos x\,I_{\varrho+n}(x)+\sin x\,I_{\varrho+n+1}(x)\},\end{aligned}\right\}\quad(21)$$

$$\left.\begin{aligned}&\int_0^x I_{\varrho-m}(\alpha)\,\alpha^{\varrho+m}\sin(2x-\alpha)\,d\alpha = \\ &= m!\,2^{\varrho+m+1}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho}{m-n}}{n!}\cdot\frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1}\{\sin x\,I_{\varrho+n}(x)+\cos x\,I_{\varrho+n+1}(x)\},\end{aligned}\right\}\quad(22)$$

$$\left.\begin{aligned}&\int_0^x I_{\varrho-m}(\alpha)\,\alpha^{\varrho+m}\cos(2x-\alpha)\,d\alpha = \\ &= m!\,2^{\varrho+m+1}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho}{m-n}}{n!}\cdot\frac{\left(\frac{x}{2}\right)^{\varrho+n+1}}{2\varrho+2n+1}\{\cos x\,I_{\varrho+n}(x)-\sin x\,I_{\varrho+n+1}(x)\},\end{aligned}\right\}\quad(23)$$

Vervangen we in (9') tot en met (15) ϱ door $\varrho+n$ en vermenigvuldigen we daarna beide leden met $\dfrac{(-1)^n\binom{m}{n}}{2^{\varrho+n-\frac{1}{2}}\,\Gamma(\varrho+n-m+\frac{1}{2})}$, dan volgen, na sommatie over n van 0 tot m, onder toepassing van V, zoo hierin vooraf ϱ vervangen wordt door $\varrho-\frac{1}{2}$, in het linkerlid onder het integraalteeken, voor $R(\nu)>-\frac{1}{2}$, $R(\varrho)>-\frac{1}{2}$ en $m\geqq 0$:

$$\left.\begin{aligned}&\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\,I_{\varrho-m-\frac{1}{2}}(\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\,\alpha^{\varrho+m+\frac{1}{2}}\,d\alpha = \\ &= \frac{\Gamma(\nu+1)\,m!\,2^{\nu+\varrho+m}}{\sqrt{\pi}}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho-\frac{1}{2}}{m-n}}{n!}\cdot\frac{\Gamma(\varrho+n+1)}{\Gamma(\nu+\varrho+n+2)}\left(\frac{x}{2}\right)^{\nu+\varrho+n+\frac{1}{2}}\{xI_{\nu+\varrho+n-\frac{1}{2}}(x)+I_{\nu+\varrho+n+\frac{1}{2}}(x)\},\end{aligned}\right\}\quad(24)$$

$$\left.\begin{aligned}&\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\,\alpha^{\varrho+m+\frac{1}{2}}\cos(x-\alpha)\,d\alpha = \\ &= m!\,2^{\varrho+m-\frac{1}{2}}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho-\frac{1}{2}}{m-n}}{n!}\cdot\frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1}\{x\,I_{\varrho+n-\frac{1}{2}}(x)+I_{\varrho+n+\frac{1}{2}}(x)\},\end{aligned}\right\}\quad(25)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\,\alpha^{\varrho+m+\frac{1}{2}}\sin(x-\alpha)\,d\alpha = m!\,2^{\varrho+m+\frac{1}{2}}\sum_{n=0}^{m}\frac{(-1)^n\binom{\varrho-\frac{1}{2}}{m-n}}{n!}\cdot\frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{3}{2}}}{\varrho+n+1}\,I_{\varrho+n+\frac{1}{2}}(x),\quad(26)$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\,\alpha^{\varrho+m+\frac{1}{2}}\sin\alpha\,d\alpha =$$

$$= m!\,2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x\sin x\, I_{\varrho+n-\frac{1}{2}}(x) + (\sin x - x\cos x)\, I_{\varrho+n+\frac{1}{2}}(x)\},$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(\alpha)\,\alpha^{\varrho+m+\frac{1}{2}}\cos\alpha\,d\alpha =$$

$$= m!\,2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x\cos x\, I_{\varrho+n-\frac{1}{2}}(x) + (\cos x + x\sin x)\, I_{\varrho+n+\frac{1}{2}}(x)\},$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(x)\,\alpha^{\varrho+m+\frac{1}{2}}\sin(2x-\alpha)\,d\alpha =$$

$$= m!\,2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x\sin x\, I_{\varrho+n-\frac{1}{2}}(x) + (\sin x + x\cos x)\, I_{\varrho+n+\frac{1}{2}}(x)\},$$

$$\int_0^x I_{\varrho-m-\frac{1}{2}}(x)\,\alpha^{\varrho+m+\frac{1}{2}}\cos(2x-\alpha)\,d\alpha =$$

$$= m!\,2^{\varrho+m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{\varrho-\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{\varrho+n+\frac{1}{2}}}{\varrho+n+1} \{x\cos x\, I_{\varrho+n-\frac{1}{2}}(x) + (\cos x - x\sin x)\, I_{\varrho+n+\frac{1}{2}}(x)\}.$$

Substitutie van $\varrho = m+\frac{1}{2}$ in (16) tot en met (23) geeft na eenige herleiding de volgende formules, geldig voor $R(\nu) > -\frac{1}{2}$ en $m \geqq 0$:

$$\int_0^x I_\nu(x-\alpha)\,(x-\alpha)^\nu\,\alpha^{2m}\sin\alpha\,d\alpha =$$

$$= m!\,\Gamma(\nu+\tfrac{1}{2})\,2^{\nu+2m} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+m+n+1} I_{\nu+m+n+1}(x), \tag{31}$$

$$\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\,\alpha^{2m}\sin\alpha\,d\alpha =$$

$$= m!\,\Gamma(\nu+\tfrac{1}{2})\,2^{\nu+2m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} I_{\nu+m+n+\frac{3}{2}}(x), \tag{32}$$

$$\int_0^x \alpha^{2m} \cos(2\alpha - x)\, d\alpha =$$

$$= \frac{x^{2m+1}}{2m+1} \cos x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{3}{2}}(x), \tag{33}$$

$$\int_0^x \alpha^{2m} \sin(2\alpha - x)\, d\alpha =$$

$$= -\frac{x^{2m+1}}{2m+1} \sin x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{1}{2}}(x), \tag{34}$$

$$\int_0^x \alpha^{2m} \cos 2\alpha\, d\alpha = \frac{x^{2m+1}}{2m+1} - m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) - \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{35}$$

$$\int_0^x \alpha^{2m} \sin 2\alpha\, d\alpha = m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) + \sin x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{36}$$

$$\int_0^x \alpha^{2m} \cos 2(x-\alpha)\, d\alpha = \frac{x^{2m+1}}{2m+1} \cos 2x + m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) + \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{37}$$

$$\int_0^x \alpha^{2m} \sin 2(x-\alpha)\, d\alpha = \frac{x^{2m+1}}{2m+1} \sin 2x - m!\, 2^{2m+1} \sqrt{\pi} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) - \sin x\, I_{m+n+\frac{3}{2}}(x)\}. \tag{38}$$

Substitutie van $\varrho = m - \frac{1}{2}$ in (16) tot en met (23) geeft, na vervanging van m door $m+1$ en eenige herleiding, de volgende formules, geldig voor $R(\nu) > -\frac{1}{2}$ en $m \geqq 0$:

$$\int_0^x I_\nu(x-\alpha)(x-\alpha)^\nu \alpha^{2m+1} \cos\alpha\, d\alpha = (m+1)!\,\Gamma(\nu+\tfrac{1}{2})\,2^{\nu+2m+1} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+m+n+1} I_{\nu+m+n+1}(x), \tag{3}$$

$$\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)(x-\alpha)^{\nu+\frac{1}{2}} \alpha^{2m+1} \cos\alpha\, d\alpha = (m+1)!\,\Gamma(\nu+\tfrac{1}{2})\,2^{\nu+2m+\frac{3}{2}} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} I_{\nu+m+n+\frac{1}{2}}(x), \tag{4}$$

$$\int_0^x \alpha^{2m+1} \sin(2a-x)\, d\alpha = = \frac{x^{2m+2}}{2(m+1)} \sin x - (m+1)!\,2^{2m+2}\sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{3}{2}}(x), \tag{41}$$

$$\int_0^x \alpha^{2m+1} \cos(2\alpha-x)\, d\alpha = = -\frac{x^{2m+2}}{2(m+1)} \cos x + (m+1)!\,2^{2m+2}\sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} I_{m+n+\frac{1}{2}}(x), \tag{41}$$

$$\int_0^x \alpha^{2m+1} \sin 2\alpha\, d\alpha = (m+1)!\,2^{2m+2}\sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) - \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{43}$$

$$\int_0^x \alpha^{2m+1} \cos 2\alpha\, d\alpha = -\frac{x^{2m+2}}{2(m+1)} + (m+1)!\,2^{2m+2}\sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) + \sin x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{44}$$

$$\int_0^x \alpha^{2m+1} \sin 2(x-\alpha)\, d\alpha = \frac{x^{2m+2}}{2(m+1)} \sin 2x - (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\sin x\, I_{m+n+\frac{1}{2}}(x) + \cos x\, I_{m+n+\frac{3}{2}}(x)\}, \tag{45}$$

$$\int_0^x \alpha^{2m+1} \cos 2(x-\alpha)\, d\alpha = -\frac{x^{2m+2}}{2(m+1)} \cos 2x + (m+1)!\, 2^{2m+2} \sqrt{\pi} \sum_{n=0}^{m+1} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n+1}}{n!} \cdot \frac{\left(\frac{x}{2}\right)^{m+n+\frac{3}{2}}}{m+n+1} \{\cos x\, I_{m+n+\frac{1}{2}}(x) - \sin x\, I_{m+n+\frac{3}{2}}(x)\}. \tag{46}$$

Substitutie van $\varrho = m+1$ resp. $\varrho = m$ in (24) geeft voor $R(\nu) > -\frac{1}{2}$ en $m \geqq 0$:

$$\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{2m+1} \sin \alpha\, d\alpha = m!\, \Gamma(\nu+1)\, 2^{\nu+2m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n+1)!}{\Gamma(\nu+m+n+3)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{3}{2}} \{x\, I_{\nu+m+n+\frac{1}{2}}(x) + I_{\nu+m+n+\frac{3}{2}}(x)\}, \tag{47}$$

$$\int_0^x I_{\nu-\frac{1}{2}}(x-\alpha)\,(x-\alpha)^{\nu+\frac{1}{2}}\, \alpha^{2m} \cos \alpha\, d\alpha = m!\, \Gamma(\nu+1)\, 2^{\nu+2m-\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n \binom{m+\frac{1}{2}}{m-n}}{n!} \cdot \frac{(m+n)!}{\Gamma(\nu+m+n+2)} \left(\frac{x}{2}\right)^{\nu+m+n+\frac{1}{2}} \{x I_{\nu+m+n-\frac{1}{2}}(x) + I_{\nu+m+n+\frac{1}{2}}(x)\}, \tag{48}$$

terwijl dezelfde substituties in (25) en (26) voeren tot de integralen, voorkomende in (41), (42) en (33), (34), wel is waar met uitkomsten van eenigszins anderen vorm. Door echter die verschillende uitkomsten voor dezelfde integralen aan elkaar gelijk te stellen, komen na eenige herleiding betrekkingen voor den dag, die uit V blijken te volgen voor $\varrho = m+\frac{1}{2}$ en $\varrho = m-\frac{1}{2}$.

Biochemistry. — *Factors determining the way in which neutral salts will affect the volume of the complex coacervate gelatine + gum arabic.* By H. G. BUNGENBERG DE JONG and C. V. D. MEER. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of March 28, 1942.)

1. *Introduction.*

In previous communications we have repeatedly studied the effect of salts on the volume of the complex coacervate gelatine + gum arabic. When we restrict ourselves to the effect of salts on an uncharged or practically uncharged complex coacervate (that is one in which the positive gelatine charge is compensated for by the negative charge of the gum arabic) we find the course of the curves to be such as pictured in Fig. 1 [1]).

Here occurs the so called *"double valence rule"* with regard to the concentrations with which the coacervation is completely neutralized (coacervate volume = 0). This rule, which has long been known and was discussed elsewhere, is not however the subject of this investigation, but the course of the curves in the smaller salt concentrations preceding

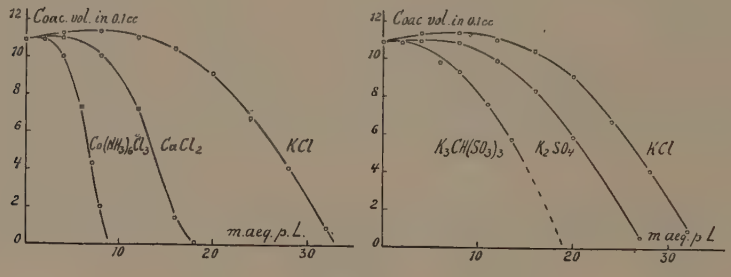

Fig. 1

the neutralization. In Fig. 1 it is seen that (apart from a slight rise above the blank level) from left to right the curves begin with a practically horizontal course, to deviate ever more downward. The order of the curves is then almost that in which they finally reach the absciss axis. In what follows we shall become acquainted with the two causes which may make the original course of these curves entirely different from Fig. 1.

2. *Methods.*

As starting point we used the same colloid preparations as in previous investigations [2]). All the experiments described here were made with the unpurified gelatine preparation as well as with the isoelectric gelatine prepared from it, the results of which were so uniform that here we shall exclusively communicate the experiments with the isoelectric gelatine.

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[1]) Fig. 1 was drawn from the experimental data necessary for a previous investigation: H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Nederl. Akad. v. Wetenschappen, Amsterdam, **45**, 59 (1942). See Table l.c. page 61 which only indicates the concentrations with which the neutralization in Fig. 1 is achieved.

[2]) Gelatine F 00 extra of the Lijm- en Gelatinefabriek „Delft" at Delft. For purification see Kolloid Beihefte, **43**, 256 (1936).

Gum arabic: gomme Senegal petite boule blanche I of ALLAND et ROBERT (Paris). This preparation was ground to a coarse powder.

The stock sols were prepared as follows: 4.5 g gum arabic + 3.75 g isoelectric gelatine + 250 cc dist. water are placed for one night in the refrigerator, the next morning they are dissolved in a waterbath at 60° for 10 minutes being now and then shaken, after which the sol is filtered through coarse filtering paper. It is used at once for an experimental series.

In sedimentation tubes we always prepared mixtures with a total volume of 27.5 cc, containing 10 cc of the above gelatine-gum arabic stock sol. Their composition was: a cc HCl 0.1 n + b cc salt solution + (17.5 —a—b) cc H_2O + 10 cc stock sol, a being usually constant and b varied, while always at the same time (or immediately after each other) the following five salts were used: $Co(NH_3)_6Cl_3$, $CaCl_2$, KCl, K_2SO_4, $K_3CH(SO_3)_3$ (representing salts types 3—1, 2—1, 1—1, 1—2, 1—3). The working temperature was 40° C. In contradistinction to the method always used previously of leaving them for some time in the thermostat — usually 1 night — and then noting the coacervate volumes, we have this time followed a different method. The coacervated mixtures were first left in the thermostat for 30 min. and then the tubes were placed in a hand centrifuge and centrifuged during 50 sec. after which they were again put in the thermostat and the coacervate volumes were noted. After this they were twice — sometimes several times — subjected to the same treatment, until the coacervate volume no longer changed. Further details concerning the method followed in § 4 will be mentioned there. First we determined the coacervate volume in the way indicated, when a was varied in the absence of added salt (b = 0), which gave the following values V for the coacervate volumes

a	V	a	V	a	V
0.3	0	1.0	13.3	1.7	7.6
0.4	1.1	1.1	13.1	1.8	5.7
0.5	5.4	1.2	12.9	1.9	3.4
0.6	9.2	1.3	12.7	2.0	1.3
0.7	10.8	1.4	12.0	2.1	0
0.8	12.0	1.5	10.7		
0.9	13.0	1.6	9.6		

Graphically (by constructing a bisecting line) we found that the maximal coacervate volume is near a = 1.06 cc, 0.1 n HCl. As shown by previous experiences the reversal of charge of the complex coacervate must be at or near this point, so that with values of a which are considerably below 1.06 the complex coacervates are certainly strongly negative, with values of a which are considerably higher than 1.06 they are certainly strongly positive.

3. *Effects of salts on a coacervate of strongly negative, resp. strongly positive charge.*

A. *Experiments.*

Three experimental series were begun with stock sols which were new-prepared for each series, a always being constant in the mixtures, namely 0.5, 1.06 resp. 1.8. In order to make the quantities of HCl in these series as reproducible as possible, we started from 0.01 n resp. 0.02 n HCl. owing to which a could be taken 10 resp. 5 times greater. b was varied with the five salts mentioned in 2. so that the effect of these salts was obtained to a final concentration of ca 10 m aeq. p. l.

The results of these three experimental series are pictured in Fig. 2, in which A refers to a coacervate of strongly negative charge (a = 0.5), B to a practically uncharged coacervate (a = 1.06) and C to a coacervate of strongly positive charge (a = 1.8). The course of the curves in Fig. 2B corresponds to that in Fig. 1 apart from a slight rise here and there above the blank level, to which we will refer again in 4, the general

tendency of the curves is a downward one. In this concentration section of 0—11.4 m eaq. the order of the "double valence" rule is reflected in the curves:

$$3-1 > 2-1 > 1-1$$

$$1-3 > 1-2 > 1-1$$

as we find it in Fig. 1 for the total neutralization. Entirely different is the course of the curves in the area of the small concentrations with the coacervate of strongly negative, resp. strongly positive charge. In the combinations of curves the curves follow each other in the order of the so-called *continuous valence* rule

$$3-1 \ldots 2-1 \ldots 1-1 \ldots 1-2 \ldots 1-3$$

It is noteworthy that from top to bottom in Fig. C this order is the reverse of the one in Fig. 2A.

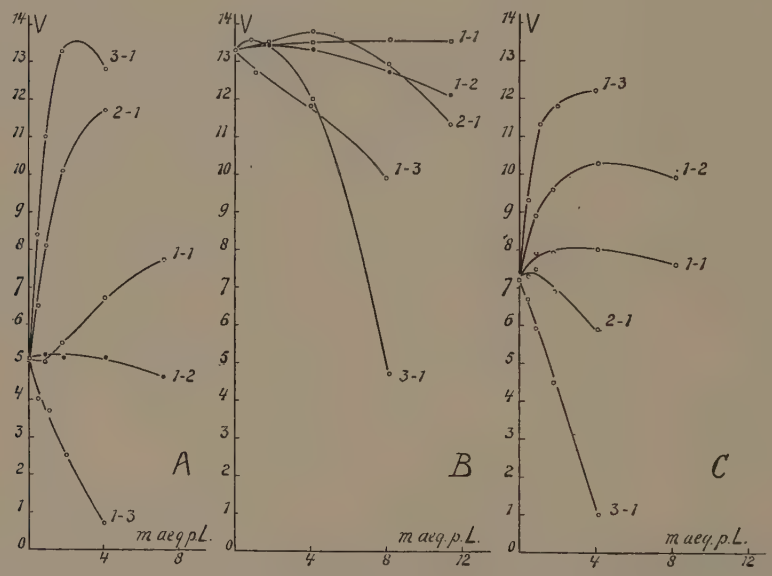

Fig. 2

B. *Discussion.*

First we want to point out that the course of the curves in Fig. 2A and Fig. 2C is not entirely unexpected, as this might practically be foreseen from the results of a previous investigation by one of us together with E. G. HOSKAM [1]). Compare Fig. 2 of the communication mentioned from which it follows that the coacervate volume of a negative coacervate is increased by 6 m aeq. $C_0(NH_3)_6Cl_3$, decreased by 6 m aeq. $K_3CH(SO_3)_3$, whereas the reverse is true of a positive coacervate.

But theoretically the course of the curves of Fig. 2A and 2C is not surprising either, as in our first detailed communication about complex coacervation [2]) it was already foreseen that under certain circumstances neutral salts can promote complex coacervation. This will be the case when with a given pH the mixing proportion of the two colloids

[1]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Nederl. Akad. v. Wetenschappen, Amsterdam, **45**, 59 (1942).

[2]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beihefte, **43**, 143 (1935); **43**, 213 (1936), see p. 188 in the publication named first.

deviates considerably from the mixing proportion of optimal coacervation belonging to that pH. The degree of coacervation (= ... fraction of the gelatine and gum arabic in the total system present in the coacervate) is then much smaller than it can be optimally. When, for instance, in the total system there is a relative excess of gum arabic (in our case with a = 0.5, so in Fig. 2A), the coacervation degree can rise on the addition of small concentrations of a salt (e.g. 3—1), the cation of which screens the charge of the arabinate colloid anion to a much greater extent than does the anion the charge of the gelatine colloid cation.

In the first three terms of series 3—1 2—1 1—1 1—2 1—3 the screening of the arabinate colloid anion decreases from left to right with equal concentration, whereas in the last three terms the screening of the gelatine colloid cation increases from left to right.

So it is to be expected, that the improvement of the complex coacervation originating from 3—1 will decrease in the following terms and will turn into a deterioration of the coacervation. When on the other hand there is in the total system a relative shortage of gum arabic (in our case with a = 1.8, so in Fig. 2C), an improvement may be expected from 1—3, while 1—2 will cause improvement to a less extent, whereas in the following terms 1—1 2—1 3—1 there will be deterioration of the coacervation.

So this accounts for the order of the salts in the curve bundles of Fig. 2A and 2C.

Practically one would also expect that the salts which in Fig. 2A cause the coacervate volume to increase (resp. decrease), will cause a decrease (resp. increase) of the volume in Fig. 2C. This expectation is approximately, but not entirely fulfilled as in Fig. 2A the course of the curve for 1—2, but in Fig. 2C that of the curve for 1—1 is almost horizontal. But in pronouncing this expectation we took it for granted that the degree of screening of the colloid ions is independent of the pH, which is probably not the case. In Fig. 2A and 2C the pH does indeed differ.

4. *Effect of KCl on the coacervate volume of the optimal coacervate after partial removal of the equilibrium liquid.*

The increase of the coacervate volume, which in Fig. 2A (coacervate of strongly negative charge) and in Fig. 2C (coacervate of strongly positive charge) is caused by some salts, was based on an increase of the degree of coacervation. So it will be clear that with optimal coacervation (a = 1.06) added salts can at most more or less disturb the compensation of charge which is already completely present, in consequence of which the coacervation degree will decrease. This applies to 3—1 and 2—1, which act positivizing and also to 1—2 and 1—3 which act negativizing. But since as was shown in a previous investigation [1]) KCl (1—1) practically does not affect the compensation of charge present with optimal coacervation, this salt is eminently suitable for a closer investigation of the other factors determining the coacervate volume.

In Fig. 1 and 2B we see that the curve for KCl at first rises very little, showing a strong decline after 10 m aeq. It would seem as if there were two opposite influences at work, the first of which (increasing the volume) in small concentration is a little stronger than the second (decreasing in volume), whereas the latter becomes stronger as the KCl concentration increases.

Now it is not difficult to pronounce an opinion as to the nature of the two separate influences. Theoretically it may be expected that in consequence of the screening of the charge of the two colloid ions by the ions of the added salt (KCl), the effective mutual attraction of the colloid ions in the coacervate must decrease, i.e. the water percentage of the coacervate must increase, the colloid percentage will decrease. If the coacervation degree did not change on the addition of KCl we should have to expect an increase of

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[1]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proceedings Nederl. Akad. v. Wetensch., Amsterdam, **45**, 59 (1942).

the coacervate volume with KCl. The shape of the KCl curve in Fig. 1, resp. 2B can then be accounted for, if at the same time the coacervation degree falls (the colloid percentage of the equilibrium liquid rises or, in other words, the "soluibility" of the coacervate increases) and that in such a way that this decrease is comparatively slight in the small concentrations, but is accelerated with higher KCl concentration. In accordance with this are the results of previous measurements [1]) of the coacervate volume and dryweight determinations of coavervate and equilibrium liquid, from which we take the results at 40°.

KCl m aeq. p. L.	Coacerv. vol. in cc	% Dryweight Equil. Liquid	% Dryweight Coacervate
0	3.68	0.50	13.20
5	3.76	0.69	12.02
10	3.68	0.96	10.90
20	2.58	1.36	8.84

If this interpretation of the course of the KCl curve in Fig. 1 and 2B is correct, we can also foresee how we shall have to conduct the experiments so that KCl will cause a considerable increase of the coacervate volume. So we must see to it that the greatly increased "soluibility" of the coacervate will be less evident in the final result. This will be the case when we take care that the equilibrium liquid itself will have a much smaller volume than usual. This can be achieved in the following way: according to the directions of 2 we first prepare in sedimentation tubes a series of mixtures without salt (b = 0) with the optimal coacervation (a = 1.06).

So after centrifuging there is in each tube a thin coacervate layer which is noted (V_0) and a large equilibrium layer (as the total volume is 27.5 cc and the coacervate layer ca 1.36 cc the volume of the equilibrium layer is ca 26.1 cc). From each time two sedimentation tubes we now remove with pipettes the same quantity of equilibrium liquid, then adding to one tube a certain volume of KCl 0.5 n and to the other tube the same volume of dist. water.

The quantities of equilibrium liquid removed were successively 0, 5, 10, 15, 20, 22.5 and 25 cc. The volumes of KCl added, resp. of dist. water were 1.1, 0.9, 0.7, 0.5, 0.3, 0.2 and 0.1 cc, which caused the final concentrations of KCl in all the tubes to be the same (19.2 m aeq. p. l). After mixing and centrifuging to constant volume the coacervate volumes were again noted (V_1). The following table gives the results:

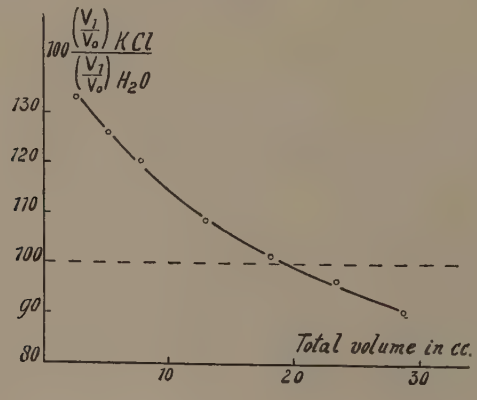

Fig. 3

[1]) H. G. BUNGENBERG DE JONG, E. G. HOSKAM and L. H. V. D. BRANDHOF–SCHAEGEN, Proceedings Nederl. Akad. v. Wetensch., Amsterdam, **44**, 1104 (1941).

In order to eliminate as much as possible the influence of sources of errors in the final result, the values of V_1/V_0 were calculated for each tube separately and the values

Final volume	Added	V_0 In 0.1 cc	V_1 In 0.1 cc	$100\frac{V_1}{V_0}$	$\frac{\left(100\frac{V_1}{V_0}\right)KCl}{\left(100\frac{V_1}{V_0}\right)H_2O}$
28.5	Dist. water	13.8	13.6	98.6	90.4
	KCl	13.7	12.2	89.1	
23.4	Dist. water	13.7	13.4	97.8	96.3
	KCl	13.9	13.1	94.2	
18.2	Dist. water	13.5	13.2	97.8	101.5
	KCl	13.7	13.6	99.3	
13.0	Dist. water	13.5	13.3	98.5	108.3
	KCl	13.5	14.4	106.7	
7.8	Dist. water	13.9	13.5	97.1	120.3
	KCl	13.7	16.0	116.8	
5.2	Dist. water	13.5	13.3	98.5	125.8
	KCl	13.4	16.6	123.9	
2.6	Dist. water	13.5	13.5	100.0	133.3
	KCl	13.5	18.0	133.3	

obtained for the mixtures with KCl were referred to those without KCl with the same final volume. In the last column of the table are given the coacervate volumes calculated thus in % of the corresponding volumes of the blanks. In Fig. 3 they are set out as functions of the final volume. The result is entirely in keeping with what was discussed above: with a large final volume of the total system 19.2 m aeq. causes a decrease of the coacervate volume, but with a small final volume it causes increase of the coacervate volume, with a final volume of 19.5 cc. 19.2 m aeq. KCl has no effect on the coacervate volume.

5. *The facts discussed in 4. referred to other salts.*

In the same way as in the previous section we examined the other salts, namely as functions of the concentration. We always removed a constant quantity (25 cc) of equilibrium liquid, centrifuged again and noted the coacervate volume (V_0) after which we added 2.5 cc salt solution (resp. 2.5 cc dist. water to the blank series), after which

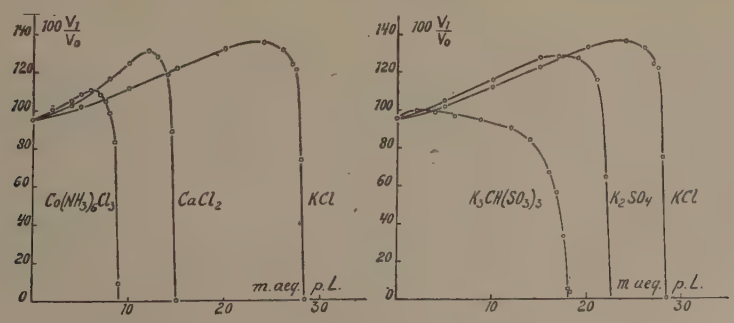

Fig. 4

we mixed and again centrifuged to constant coacervate volume (V_1). The results expressed in 100 V_1/V_0 are set out in Fig. 4. This figure should be compared with Fig. 1. Both give the effect of salts on the coacervate volume with optimal coacervation. But in Fig. 1 the total volume is 27.5 cc, in Fig. 4 only 5 cc.

Fig. 4 shows that not only KCl, but also the other salts in smaller concentrations increase the coacervate volume, before with higher concentrations the curves fall suddenly in the order of the double valence rule. What we have said in 4. concerning KCl naturally also applies to the other salts, so that the original rise of all curves is fully to be expected. Neither is it astonishing that the rising sections of the curves also follow the order of the double valence rule.

6. *Extension to coacervates with a strongly negative, resp. strongly positive charge.*

In the same way as in 5. we examined the effect of salts after the removal of 25 cc equilibrium liquid of a coacervate of strongly negative (a = 0.5) and a strongly positive (a = 1.8) charge. In the latter case at first a complication arose, owing to the subsequent dilution of the system from 2.5 to 5 cc, in consequence of which the results became irregular and were not in accordance with what was expected. This complication is probably owing to an excessive removal of HCl bound to the colloids of the equilibrium liquid. Hence they disappeared when after removing 25 cc equilibrium liquid we added 0.2 cc HCl 0.05 n to each sedimentation tube. The results of these experimental series

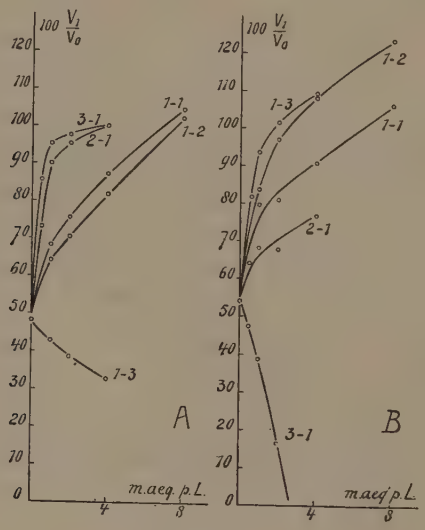

Fig. 5

with which the total volume was 5 cc, are set out in Fig. 5. When we compare these with the corresponding series with a total volume of 27.5 cc (Fig. 2A and 2C) we do not — at first sight — note any striking differences. Here too the curves in the bundle succeed each other according to the continuous valence rule, and the order in Fig. 5A is the reverse of that in Fig. 5B.

So the changes of the coacervate volume are here again determined in the first place by the factor discussed in 3. That in spite of this the coacervate-volume-increasing-effect of a small total volume is felt even here is, however, accounted for by the fact that in Fig. 2A and 2C only 3, resp. 2 out of the 5 curves rise, but here in both cases 4 out of the 5.

Summary.

1. Neutral salts affect the coacervate volume by a: a change of the degree of coacervation, b: a change of the water percentage of the coacervate.

2. With optimal coacervation neutral salts decrease the coacervate volume according to the double valence rule, because in the final result the decreasing effect of the decreasing coacervation degree surpasses the increasing effect in consequence of the increase of the waterpercentage of the coacervate.

3. On preliminary removal of a sufficient quantity of equilibrium liquid, neutral salts increase the coacervate volume in the area of small concentrations, because here the reverse of 2. is applicable. With higher concentrations however the decreasing coacervation degree predominates, which causes a very rapid decrease of the coacervation volume.

4. With a coacervate of strongly negative, resp. positive charge, certain salts (with polyvalent cations in the negative, with polyvalent anions in the positive coacervate) in smaller concentrations can cause an increase of the coacervation degree. The salts then arrange themselves in the order of the continuous valence rule.

5. On preliminary removal of a sufficient quantity of equilibrium liquid what has been said in 4. applies, only it can be seen from the location of the curve bundles that the coacervation-volume-increasing-effect of 3. also plays a part.

Leiden Laboratory for Medical Chemistry.

Biochemistry. — *Behaviour of microscopic bodies consisting of biocolloid systems and suspended in an aqueous medium. VIII. Formation and properties of hollow spheres from coacervate drops containing nucleic acid.* By H. G. BUNGENBERG DE JONG and C. V. D. MEER. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of April 25, 1942.)

1. *Introduction.*

In Communication VII [1]) of this series we said that $CaCl_2$ and $Co(NH_3)_6Cl_3$ cause the formation of hollow spheres from the G + N + a coacervate of the composite coacervate drops which on pH reduction are formed in sol mixtures G : N : A = 3 : 1 : 1 (G = gelatine, N = Na-Nucleinate, A = gum arabic). In § 3 there follow further observations concerning the formation and properties of these objects, preceded in § 2 by some other ways in which hollow spheres can be formed from the G + N + a coacervate. As for colloid preparations used and general methods we refer to Communication VII.

For an explanation of the mechanism of the formation of hollow spheres by salts from the G + N + a coacervate it is important that we already find an analogous formation of hollow spheres in the G + N coacervate. In what follows we shall discuss this more in detail (§ 4). It was desirable to make a preliminary investigation of the effect of salts on the coacervate volume of the G + N coacervate, the results of which we add briefly in § 5.

2. *Some other ways in which hollow spheres may be obtained from the* G + N + a *coacervate.*

A. *Effect of* pH *reduction and* pH *increase on complex coacervate drops.*

When according to the directions of § 3 in Communication VII in the auxiliary apparatus described there, we have prepared the composite coacervate drops and reduce the pH by the gradual addition of HCl (e.g. at intervals of 15 minutes each time 5 cc HCl 0.01 N), we note some phenomena which are not very striking: the G + A + n wall of the composite coacervate drops decreases in volume, to disappear finally, while perhaps the vacuolization of the enclosed G + N + a drop increases a little. So we see that pH reduction has an effect analogous to that of KCl in Communication VII.

When we first prepare the composite coacervate drops and then increase the pH on the other hand, by gradually adding Na-acetate (e.g. every five minutes 1 cc Na-acetate 0.1 N) the volume of the G + A + n wall is also first seen to decrease. After it has disappeared entirely the remaining coacervate drop (originally the enclosed G + A + n coacervate of the composite drops) strongly vacuolizes on continued pH increase and after a stage of froth structure a typical hollow sphere is formed with a rather thin wall (Fig. 1 a—f). When the system is cooled to room temperature while under the microscope we add some granules of saccharosis, we observe invagination (Fig. 1 f—g). This disappears on dilution with water, which shows that the invagination was brought about by osmotic dehydration, the changes described in Fig. 1 c—f are analogous to the changes previously described in the complex coacervate gelatine-gum arabic. The formation of hollow spheres described here we shall not discuss further, in this place only mentioning

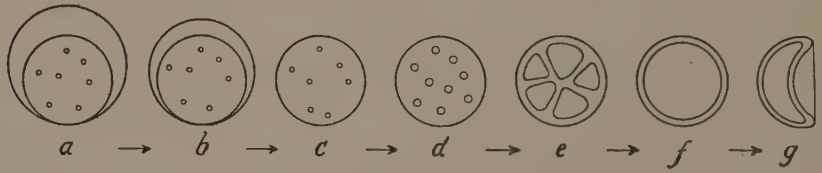

Fig. 1.

[1]) H. G. BUNGENBERG DE JONG, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 393 (1942).

the fact, that drops of the complex coacervate gelatine-nuclein acid also change into hollow spheres, by increasing the pH which are also invaginated after gelatination, with cane sugar.

B. *Hollow sphere formation in consequence of dilution with much dist. water.*

Here follow some observations made several times, and very interesting from a morphological point of view, presenting many problems the solutions of which we have not yet attempted. When in the usual way (10 cc buffer + 10 cc dist. water + 5 cc stock sol G : A : N = 3 : 1 : 1) we have prepared the system of coexisting coacervate drops, adding a great quantity (e.g. 50 cc or more) dist. water, we sometimes see that the G + N + a coacervate is gradually lifted from the edge of the composite drop by a cavity, although locally the contact persists (Fig. 2 a—b). In other cases we get the impression that the

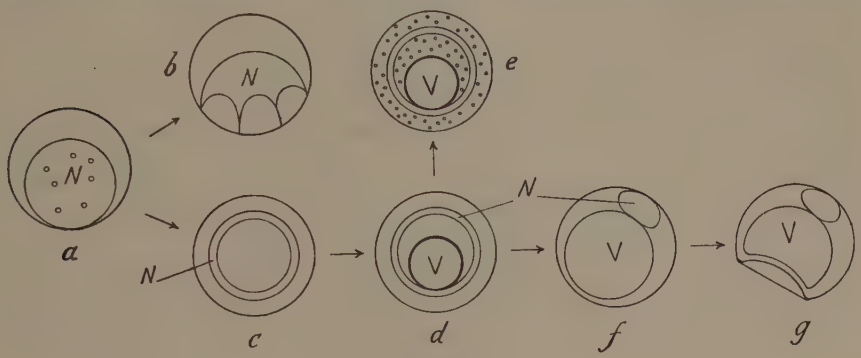

Fig. 2.

enclosed G + N + a coacervate has lost contact with the outside of the drop enclosed, gradually passing as an independent body into a hollow sphere with a rather thick wall (Fig. 2 a—c).

Here we have frequently observed that within the cavity in the G + N + a coacervate there is formed another cavity much more sharply circumscribed. By focussing the microscope at different depths we can observe that this second cavity is optically not so dense as the one in the G + N + a coacervate, so that here there is a *vacuole* (Fig. 2d V). The rest of the cavity in the G + N + a coacervate must then be filled with newly formed G + A + n coacervate. This is also indicated by the same behaviour of this cavity and the enveloping G + A + n coacervate on pH reduction, when granulation arises, see Fig. 2e (formation of small, new G + N + a coacervate drops).

Finally we note that the morphologic condition of d does not represent an equilibrium, as when a preparation is long left to itself in the auxiliary apparatus at 40° the condition of Fig. 2f is gradually reached, the entire contents of the hollow G + N + a drop being broken through, the two G + A + n coacervates having united, whereas the vacuole is still enclosed in the G + A + n coacervate. On cooling and after the addition of cane sugar the G + A + n wall invaginates (Fig. 2g), becoming round again when dist. water is added.

Here we make brief mention of the remarkable morphologic structure of the bodies of Fig. 2f, which has many points in common with some naked plant cells (occurring for instance in the pulp of ripe berries). The morphologic constellation present, is otherwise the same as occurs in the prismatic cells of celloidin membranes [1]), which have much in common with the morphologic structure of normal protoplasts enclosed in cells. The only difference is that here we find this constellation without an enclosing wall substance.

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[1]) H. G. BUNGENBERG DE JONG, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 76 (1942).

3. *Further details concerning the formation of hollow spheres from the* G + N + a *coacervate by the addition of salts.*

A. *Effect of an excess of salts 2—1 and 3—1 on the hollow spheres thus formed.*

In Communication VII we described how $CaCl_2$ and $Co(NH_3)_6Cl_3$ cause the formation of hollow spheres from the G + N + a coacervate. With slightly greater concentrations there is a new effect, namely that the originally beautifully homogeneous pictures become less fine, sometimes even granular (markedly with $Co(NH_3)_6Cl_3$). We get the impression that the G + N + a coacervate assumes a more membrane-like character. It has often a corrugated surface, becomes blurred and granular. But also in the equilibrium liquid a finely granular precipitation is formed. The cause of this morphologic degeneration is to be found in the formation of the autocomplex system: bi- resp. trivalent cation + nucleinate colloid anion.

Hence nucleinate sols show a similar granular floculation with bi- and trivalent cations, but not with salts of type 1—1, 1—2 or 1—3.

B. *Specific salt effects on the formation of hollow spheres.*

In Communication VII we have seen that with regard to $CaCl_2$, $Co(NH_3)_6Cl_3$ acts equally strongly already at smaller concentrations, while KCl does not cause the formation of hollow spheres (at least as long as there are composite coacervate drops). So to the activity of the salts decreasing from left to right the following valence rule applies:

$$3—1 > 2—1 > 1—1$$

It has appeared that hollow spheres may also be formed owing to KCl, K_2SO_4 and $K_3CH(SO_3)_3$, although this only happens with concentrations at which the surrounding G + A + n coacervate has already been neutralized. The salts mentioned then arrange themselves in the valence rule of the anions $1—3 > 1—2 > 1—1$. We also investigated by the same method if there are any differences between $MgCl_2$, $CaCl_2$, $SrCl_2$ and $BaCl_2$. Here we found that the last two have the weakest effect, but we could not obtain any certainty as to the order of Sr and Ba. The order of the four cations was found to be as follows (according to their activity decreasing from left to right concerning the formation of hollow spheres)

$$Ca > Mg > Sr, Ba$$

The order of the ions agrees with the order found by O. BANK and E. G. HOSKAM [1]) in their investigations of the specific effect of ions on the neutralization of the complex coacervate gelatine — nucleic acid (G + N).

$$Ca > Mg > Sr > Ba,$$

While as regards the general valence effect it was also found [2]) that $3—1 > 2—1 > 1—1$ and $1—3 > 1—2 > 1—1$. When to this we add the observation published in another Communication, that KCl, added to a coacervated G + N system causes the formation of hollow spheres [3]) then it is very likely that

a. The problem of the formation of hollow spheres from the G + N + a coacervate by the addition of salts is fundamentally the same as in the G + N coacervate,

b. The formation of hollow spheres in the G + N + a coacervate and in the G + N coacervate is intimately connected with the processes attending the neutralization of the complex coacervate by added salts.

It is for these reasons that we have further studied the G + N coacervate and in the following section (§ 4) we shall first see if what has been said in a. applies, while in § 5 we shall further discuss the supposition in b.

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[1]) O. BANK and E. G. HOSKAM, Protoplasma **34**, 188 (1940).

[2]) O. BANK and E. G. HOSKAM, loc. cit. (Valence of cations), H. G. BUNGENBERG DE JONG and ONG SIAN GWAN, Biochem. Z. **221** (1930).

[3]) H. G. BUNGENBERG DE JONG, O. BANK and E. G. HOSKAM, Protoplasma **34**, 30 (1940) p. 41.

4. *Formation of hollow spheres from the* G + N *coacervate by the addition of salts.*

A. *Method.*

With the same pH value the water percentage of gelatine-nucleic acid coacervates is considerably below that of gelatine-gum arabic coacervates (the consequence of greater density of charge of the nucleinate colloid anion). The result is that when one wants to prepare G + N coacervates which are sufficiently liquid and suitable for morphological investigation, the pH must be chosen considerably higher. Whereas for the G + A coacervates pH 3.7 is a suitable pH, a considerably higher pH is preferable with G + N coacervates. Moreover, in the complex combination G + N there is an additional complication on acidification in the formation of nucleic acid. As we shall see in § 5 these disturbances occur already below pH 3.8. In what follows we worked with pH 4.4, when we could easily obtain coacervate drops which were beautifully homogeneous and of sufficient size for morphological investigation.

We always started from a stock sol consisting of 9 gram gelatine F00 extra + 3 gr.Na nucleinate + 108 cc H_2O. After the mixture has been left one night in the refrigerator it is dissolved by heating and used at once in experiment (for preparations and the method see Communication VII).

In the auxiliary apparatus we place at 50°:

18.2 cc dist.water + 1.8 cc HCl 0.1 N and then 5 cc of the above stock sol (these quantities are in agreement with optimal coacervation see § 5). After 5 min. the coacervate drops are sufficiently large and homogeneous, after which the study of the effect of added substances can set in. When 10 cc dist.water is added, no morphologic changes take place.

So when with an added salt solution, usually a much smaller volume than 10 cc, we do note changes, this is to be ascribed to the salt and not to the water in which it is dissolved. In examining the salts the following method may be followed.

We first prepare a coacervated system, after 5 min. we add a smaller quantity (e.g. 1 resp. ½ cc $CaCl_2$0.1N) salt solution, observe possible morphologic changes for 5 min. add the same quantity of salt solution, observe for another 5 min. thus continuing the intermittent addition of the salt solution.

A variant of this method is the addition of salt solution every 30 min. instead of every 5 minutes. In what follows the two methods are referred to as the 5 minute and the half hour method.

B. *Effect of the cation and anion valence on the change of the* G + N *coacervate drops into hollow spheres.*

By the 5 minute and the half hour method we first investigated the effect of the valence of the cation and the anion. We found that in sufficient concentrations *all the salts examined change the coacervate drops into hollow spheres.*

The changes are similar to those described in Communication VII for the effect of $CaCl_2$ and $Co(NH_3)_6Cl_3$ on the G + N + a coacervate, namely first increased vacuolization, then coalescence of the vacuoles to few vacuoles, the drop sometimes assuming the character of a froth mass, after which the hollow sphere stage is reached.

The results of the 5 minutes and the half hour method are shown in Table I.

In column 3 are stated the values of the quantities added every 5 resp. 30 minutes of the salt solutions mentioned in column 2 (to $K_3CH(SO_3)_3$ we first added solid salt till 15,20 resp. 22 m.aeq. final concentration and after ca. 10 min. we continued by adding the 40 m.aeq. salt solution).

The final concentrations which are just too low for the maximal effect are found in column 4, they precede those which are sufficient.

By the two methods we find the orders:

$$Co(NH_3)_6Cl_3 > CaCl_2 > KCl$$
$$K_3CH(SO_3)_3 > K_2SO_4 > KCl$$

TABLE I.

Addition of salt to 25 cc coacervated G + N system, which are not yet and just (underlined) sufficient to change practically all the coacervate drops into hollow spheres.

	Salt solution		Addition in cc	Final conc. in M.AEQ.pL.
5 min. method	$Co(NH_3)_6Cl_3$	0.1 N	0.5 — 0.5 — 0.1 — **0.1**	4.2 — 4.6
	$CaCl_2$	0.1 N	1 — 1 — 0.5 — 0.5 — 0.5 — **0.5**	12.5 — 13.8
	KCL	0.5 N	0.5 — 0.5 — 0.5 — 0.5 — **0.1**	37.0 — 38.7
	K_2SO_4	0.5 N	0.5 — 0.5 — 0.5 — 0.2 — **0.1**	31.8 — 33.6
	$K_3CH(SO_3)_3$ 22 m.aeq.	0.04N	**1.0**	22.0 — 23.7
½ hour method	$Co(NH_3)_6Cl_3$	0.1 N	0.7 — 0.1 — 0.1 — **0.1**	3.5 — 3.8
	$CaCl_2$	0.1 N	2.0 — 0.5 — **0.5**	9.1 — 10.7
	KCl	0.5 N	1.0 — 0.25 — 0.25 — **0.25**	28.3 — 32.7
	K_2SO_4	0.5 N	1.0 — 0.1 — **0.1**	21.1 — 22.9
	$K_3CH(SO_3)_3$ 15 m.aeq.	0.04N	0.5 — 0.5 — 1.0 — 1.0	
	$K_3CH(SO_3)_3$ 20 m.aeq.	0.04N	**1.0**	17.8 — 20.7

that is, the order in which the salts from left to right have a decreasing neutralizing effect on the complex coacervates:

$$3-1 > 2-1 > 1-1$$
$$1-3 > 1-2 > 1-1 \text{ "double valence rule".}$$

C. *Specific effect of the cations.*

In order to investigate accurately the specific effect of $MgCl_2$, $CaCl_2$, $SrCl_2$ and $BaCl_2$, we observed the morphologic changes of the following mixtures with each time 4 of these salts during 5 min. and 30 min.:

25 cc coacervated system + 2.5 cc 0.1 N salt solution
25 cc coacervated system + 3 cc 0.1 N salt solution
25 cc coacervated system + 3.5 cc 0.1 N salt solution

and we stated the order:

$$Ca > Mg > Sr, Ba$$

analogously LiCl, NaCl and KCl were compared with each other, but the differences are so slight, that we can only pronounce the supposition that the order is

$$Li > Na > K$$

D. *Other points of agreement between the* G + N *and* G + N + a *coacervates.*

With $CaCl_2$ and $Co(NH_3)_6Cl_3$ degeneration phenomena also occur in the G + N coacervate. In the 5 min. method the first degeneration symptoms coalesce approximately with the optimal formation of hollow spheres. In the 30 min., method the two are distinct, so that in a certain concentration section, sound hollow spheres may be obtained. Again the hollow spheres show lengthy invagination with cane sugar. The invagination is of short duration with salts (e.g. after the additions of some granules of KCl under the microscope). The G + N coacervate also gives hollow spheres on pH increase (NaOH) which also invaginate with cane sugar.

5. *On the mechanism of the change of* G + N + a *resp.* G + N *coacervate drops into hollow spheres through the addition of salts.*

From what was mentioned in § 4 b—d appears the great similarity in the change into hollow spheres of G + N + a coacervate drops and of G + N coacervate drops through

the addition of salts, so that we may safely assume that if we can account for the G + N coacervate, the same will apply to the G + N + a coacervate. In order to make quite sure we also tried if the addition of KCl will cause the formation of hollow spheres from the G + A coacervate. The result is vacuolization, sometimes a large central vacuole is formed, but it disappears spontaneously after 5—10 minutes, leaving only homogeneous coacervate drops. On the other hand, the walls of the hollow spheres which are formed by adding KCl to the G + N coacervate are thinner and the spheres last much longer (e.g. 6 hours at 50°), after which homogeneous coacervate drops are gradually formed again. In these respects, therefore, the hollow spheres from the G + N + a coacervate are very much like those of the G + N coacervate and unlike those of the G + A coacervate.

As regards the mechanism of the formation of hollow spheres from the G + N coacervate, the occurrence of the double valence rule and of the specific cation order: Ca > Mg > Sr, Ba (and possibly Li > Na > K) point to the intimate connection with events attending the neutralization by salts of complex coacervate G + N. In order further to support this conclusion, we made preliminary measurements at 40° of the effect of salts by the coacervate volume method, which we have often applied in our study of the G + A coacervate [1]). These tests are conducted so that they are practically comparable with the method described above: 25 cc final volume, containing 5 cc stock sol (9 G + 3 N + 108 H_2O). The coacervate volumes were noted after centrifuging to constant volume.

Fig. 3 shows the effect of added HCl. It is seen that the coacervate volume reaches

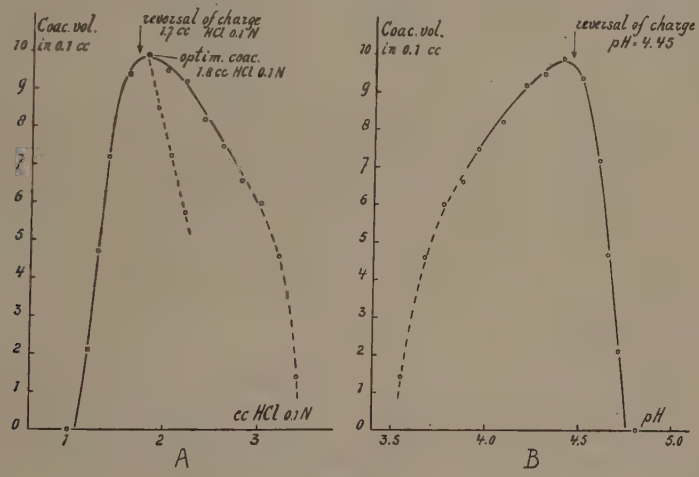
Coac. vol. in 0.1 cc
reversal of charge 1.7 cc HCl 0.1 N
optim. coac. 1.8 cc HCl 0.1 N
cc HCl 0.1 N
A
Coac. vol. in 0.1 cc
reversal of charge pH = 4.45
pH
B

Fig. 3.

a maximum at 1.8 cc HCl 0.1 N (determined graphically by constructing a bisecting line).

In Fig. 3b the coacervate volumes are set out against the pH values measured at 40°; it appears that the peak of the curve lies at pH 4.40. The optimal coacervation, according to electrophoresis measurements lies at 40° very near the reversal of charge, which we found to be with 1.7 cc HCl = pH 4.45 [2]).

The part of the curves of Fig. 3, where the equilibrium liquids as well as the coacervates were entirely clear has been drawn in full, only towards a greater quantity of HCl (resp. towards a lower pH) the equilibrium liquid (formation of nucleic acid, see § 4a) begins

[1]) e.g. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 247 (1939), **45**, 3 (1942).

[2]) Here we wish to express our thanks to Dr. H. L. BOOY for his assistance in measuring the electrophoresis velocity.

to be opalescent (pH 3.87—3.67). The coacervate is then fairly transparent still, but by still lower pH values, it becomes more and more opalescent and less transparent, like the equilibrium liquid.

Fig. 4a and 4b show the effect of neutral salts at optimal coacervation (i.e. at 1.8 cc HCl constant) [1]). Here we see the order of the salts (double valence rule) characteristic

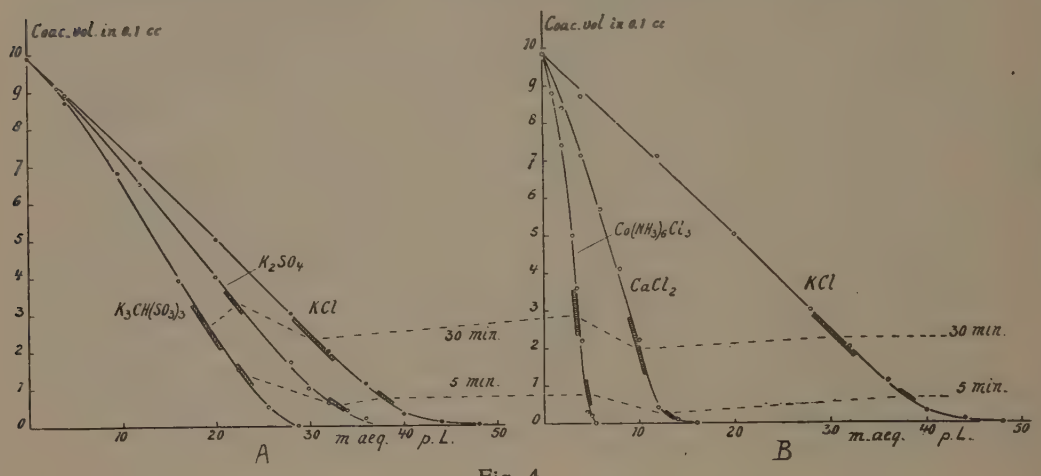

Fig. 4.

of the neutralization of complex coacervates. In Fig. 4 we have clearly marked the concentration sections extending between the two each time named concentrations in Table 1.

Indeed we see that the concentrations from which the hollow spheres are characteristically formed lie at the bottom of the coacervate volume curves (by the 5 min. method at roughly 0.1 and by the half hour method at roughly 0.25 of the original coacervate volume).

So hollow spheres do, indeed, develop as a co-effect of the neutralization of the coacervate by salts, rapidly (in 5 min.) when 0.9, slowly (in 30 min.) when 0.75 of the original coacervate volume will disappear through neutralization by salts.

The following observations are of importance for the problem of the mechanism of the formation of hollow spheres:

A. Cane sugar does not produce hollow spheres at ca. 100 m. mol final concentration. The presence of cane sugar in this concentration does not prevent the formation of hollow spheres by salts (e.g. KCl,) they also persist when afterwards cane sugar is added.

B. When through a salt a system of hollow spheres has first been formed, the extra addition of 10 cc dist.. water does not cause the spheres to disappear, but the walls become a little thicker. When, however, 10 cc 10 % amylum solubile (MERCK) is added to a system of hollow spheres formed by a salt, the hollow sphere changes in 10—15 min. into a homogeneous coacervate drop, which condition is still present after 2 hours. 10 cc 10 % dextrine has the same effect. When to the original coacervate system 10 cc 10 % amylum solubile is added first, it is no longer possible to obtain hollow spheres by the addition of KCl.

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[1]) Measurements to the left of the peak (at 1.3 cc HCl 0.1 N) and to the right of the peak (at 2.6 cc HCl 0.1 N) show a course of the curves analogous to that we lately described for the G + A coacervate (H. G. BUNGENBERG DE JONG and C. V. D. MEER, Proc. Ned. Akad. v. Wetensch., Amsterdam, **42**, 490 (1942) i.e. in the first case (negative coacervate) in *small* salt concentrations there is the order of curves (from top to bottom) 3—1 ... 2—1... 1—1...1—2...1—3 in the second case (positive coacervate) the order is reversed. In the higher salt concentrations the curves in both cases approach the absciss axis in the order of the double valence rule.

So we state a difference between a molecularly dissolved non-electrolyte (cane sugar) and a colloidally dissolved non-electrolyte [1]) (amylum, dextrine).

The former penetrates into the coacervate, the latter probably does not (at least we know this for complex coacervate G + A). Although after gelatination the wall of the hollow spheres is not easily penetrated by saccharosis, this is apparently not the case at a working temperature of 50°, according to A. But according to B. we must assume impenetrability for amylum solubile, even at 50°.

So we arrive at the conclusion that according to B. we can prevent the formation of hollow spheres, resp. that we can change hollow spheres that have formed already into homogeneous coacervate drops, by giving a certain colloid-osmotic pressure to the surrounding equilibrium liquid. Therefore it seems probable to us that the hollow spheres are also formed in consequence of colloid-osmotic overpressure of the contents of little vacuoles, formed primarily as a result of added salts. The question remains why with the G + N coacervate and to a less extent with the G + A coacervate, there is temporarily a colloid osmotic overpressure of the contents of the vacuoles with regard to the medium surrounding the coacervate drops. We tentatively offer the following explanation, based on the rapid reaction to changes of the medium of the G + A coacervate and on the slow reaction of the G + N coacervate.

This can indeed in principle account for the above: the contents of the vacuoles will have assumed the higher colloid percentage of the equilibrium liquid belonging to the salt concentration long before this is the case with the great volume of the medium surrounding the coacervate drops. In this period there is colloid-osmotic overpressure of the vacuole contents, so that they want to take up water. When this happens, however the colloid osmotic pressure of the vacuole contents is not diminished because the water taken up, at once passes again into equilibrium liquid by taking up colloids from the coacervate surrounding the vacuole.

The driving force for the inflow of water into the vacuoles, owing to which they become ever larger and coalesce so that finally hollow spheres are formed, persists until at last the surrounding medium liquid has also assumed the composition of the equilibrium liquid. Owing to the rapid reaction of the G + A coacervate to changes of the medium, the colloid osmotic non-equilibrium between primary vacuoles and surrounding medium liquid is of short duration, so that the formation of hollow spheres takes place during a short time only.

Summary.

1. Further details are given concerning the formation and properties of hollow spheres from the G + N + a complex coacervate of composite coacervate drops.

2. Added salts also change drops of the G + N complex coacervate into hollow spheres and here again we find the influence of ion valence (double valence rule) and the specific ion order: Ca > Mg > Sr, Ba.

3. Measurements of the effect of salts on the volume of the G + N coacervate show that with concentrations in which typical hollow spheres are formed the coacervate is neutralized for the greater part.

4. A provisional explanation is given of the mechanism of the formation of hollow spheres by added salts, according to which a temporarily existing colloid-osmotic non-equilibrium between medium and contens of the primary vacuoles is taken as starting point. This explanation also accounts for the fact that the formation of hollow spheres by adding salts to the G + A complex coacervate is not nearly so striking.

Leiden, Laboratory for Medical Chemistry.

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1) Amylum solubile MERCK, although it has a weak capillar electric negative charge, has a very high reciprocal hexol number and can therefore be considered practically a non-electrolyte. As a matter of fact it does not form a complex coacervate with positive gelatine.

Medicine. — *The demonstration of a disordered lung function by means of a blood-gasanalysis.* By C. DIJKSTRA[1]). (Formerly assistant Sanatorium Berg en Bosch.) (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of April 25, 1942.)

Beside the examination of the diminution of the volumina of the lungs and of the ventilation reserves, caused by different pathological changes in the lungs, it is also important to investigate the transport of respiration gasses which is, perhaps, disordered. As carbonic acid diffuses more easily than oxygen, it must be expected that a disorder of the oxygen absorption in the blood will occur sooner than an impediment of the discharge of the carbonic acid. So the examination of the function of the lungs by means of an analytical demonstration of the bloodgas can be limited to the investigation of the transport of oxygen.

The most direct and exact way to demonstrate disorders in the oxygen transport from the alveolar air to the blood is the determination of the oxygen content of the arterial blood. If the oxygen content is decreased the existence of a disorder is proved. By further examination the cause of the disorder can be traced. Here we have to distinguish between *disordered ventilation* (either by pathological changes in the respiratory centrum or by changes in the thorax of the lungs) and a *disordered diffusion* in the lung parenchym.

Formerly it was very difficult to determine the oxygen content of the arterial blood as a rather large quantity of blood was necessary for this analysis so that an arterial punction had to be done. This made a serial examination nearly impossible. In recent years, however, the oxygen content in small quantities of arterial blood could be determined. We herefore used the so-called haemoxymeter of BRINKMAN (Groningen). The capillary blood (from finger tip or ear) suffices as it has become apparent that this is nearly completely similar to arterial blood (VERZAR and KELLER). LUNDSGAARD and MÖLLER found that after warming the hand, the capillary blood from a finger has an oxygen content of 97 per cent, which completely agrees with the oxygen content of the arterial blood. It even appeared that under these circumstances blood from the veins of the hands was completely (i.e. to 97 per cent) saturated with oxygen (GOLDSCHMIDT and LIGHT).

So a sufficient warming of the hand certainly causes the capillary blood of the finger tip to have the same oxygen content as the arterial blood. These facts make it possible to study the oxygen transport through the lungs in a simple way. A decreased oxygen content of the arterial blood points to a disorder in the oxygen transport.

On the other hand it cannot be said that a normal oxygen content of the arterial blood points to a transport of oxygen which is *not* disordered. DIRKEN and KRAAN demonstrated this by a special investigation. They examined the arterial blood under different circumstances, viz. first when the subject breathed the normal air of the room (this air contains about 21 per cent of oxygen) and secondly when the subject inhaled a mixture of gas, containing 17 or 15 per cent of oxygen.

In the last case the tension of oxygen in the alveolar air decreases, by which the diffusion of the oxygen to the blood is made more difficult as the difference in oxygen pressure between alveolar air and blood becomes smaller.

DIRKEN and KRAAN found that the decrease of the oxygen content of the arterial blood which appears under these circumstances, is less marked in normal individuals than in

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1) The investigations were performed in collaboration with L. BRONKHORST, medical student in Utrecht.

patients with diseases of the lungs. In this way they could demonstrate latent *disorders in the function of the lungs.*

The function experiment of DIRKEN and KRAAN can be considered as a ration test for the oxygen transport. The process of diffusion in the lungs is impeded by it. In cases where a disordered gasdiffusion was already present (by pathological affections of the lung tissue) it will be the cause of a worse arterialisation of the blood than when the lung membrane had a normal perfusion for the respiration gasses.

In a number of patients suffering from pulmonary tuberculosis, we tried to investigate if and to which extent a disordered oxygen transport was present. Our examinations are less complete than those of DIRKEN and KRAAN as we confined ourselves to the examination with air- and with 17 per cent oxygen-respiration.

From a recent publication of DIRKEN, KRAAN, OOSTINGA and WOUDSTRA it appeared that the examination gave sometimes irregular results which makes it necessary to check the results by means of an examination with 15 per cent oxygen respiration.

However, the results of our investigation, compared with those of DIRKEN c.s., agree in the main, so that it can be assumed that the above mentioned irregularities did not play a prominent part.

In table I the results of this investigation are given.

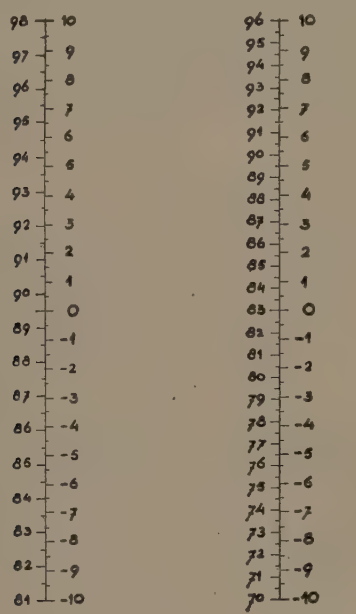
TABLE I.

Name	Group I				Name	Group II				Name	Group III			
	21 % oxygen		17 % oxygen			21 % oxygen		17 % oxygen			21 % oxygen		17 % oxygen	
	% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.		% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.		% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.
J. C.	94	5.3	89½	5	E. H.	95½	7.1	91	6.2	G. B.	*93½	4.7	85	1.6
B. D.	95	6.4	90	5.4	P. H.	95	6.4	90	5.4	B. v. B.	97	8.8	92	6.9
J. D.	95½	7.1	90	5.4	C. V.	96	7.7	90½	5.7	P. D.	95	6.4	88	3.9
J. H.	96	7.7	94	8.4	H. v. V.	95½	7.1	90½	5.7	J. v. D.	*97	8.8	89	4.6
J. M.	94½	5.9	90	5.4	H. Z.	96	7.7	90½	5.7	A. v. H.	94½	5.9	90½	5.7
N. B.	94½	5.9	91	6.2	v. H.	98	10	93½	8.1	J. L.	93	4.1	89½	5.0
H. P.	94½	5.9	90	5.4	J. W.	95	6.4	93	7.7	L. v. d. L.	*94	5.3	83	0
v. T.	94	5.3	91¼	6.6	v. P.	92½	3.6	88¼	4.2	N. D.	*93	4.1	83	0
v. d. A.	94	5.3	90½	5.7	B. P.	95	6.4	90½	5.7	G. P.	*96½	8.2	88	3.9
J. K.	96½	8.2	94½	8.8	J. v. V.	94	5.3	89½	5	H. v. d. V.	93	4.1	88½	4.2
W. F.	95	6.4	92	6.9	J. D.	94	5.3	90½	5.7	J. V.	*93¾	4.7	82	—1
C. Q.	94½	5.9	91	6.2	F. D.	95	6.4	90½	5.7	W. G.	91½	2.4	85	1.6
J. C.	96½	8.2	92½	7.3	v. H.	93	4.2	89½	5	F. P.	*88	0.2	76½	—6
C. D.	95	6.4	90½	5.7	v. G.	94½	5.9	88	3.9	P. P.	*97	8.8	88	3.9
W. C.	94	5.3	89½	5	J. B.	95	6.4	91	6.2	C. V.	*94	5.3	85	1.6
P. D.	95	6.4	90	5.4	V. M.	94	5.3	90	5.4	J. D.	89	(0.5)	83½	0.4
					W. P.	94¼	5.9	90½	5.7	L. H.	89	(0.5)	83½	0.4
										J. L.	92	2.9	85½	1.9
Average	95%	6.4	91%	6.2	Average	95%	6.4	89%	4.8	D. P.	92½	3.5	87	3.1
										P. B.	90	0.6	84	0.7
										F. D.	*93	4.1	85	1.6
										L. L.	89	(0.5)	82	—1
										J. D.	95	6.4	89	4.6
										Average	93½%	4.6	86%	2.3

To explain the table the following data are given.

The patients — exclusively cases where no collapse therapy had been applied — have been divided into 3 groups. Group I includes the cases with only slight lung affections. Group II consists of patients with rather extensive lung processes, in group III the patients suffering from very serious (often bilateral) diseases are classed.

In column 1 and 3 the percentages of the oxygen content of the arterial blood are given, respectively for 21 and 17 per cent oxygen respiration. In the other columns (2 and 4) the figures for comparison are given. The notion *"comparative figure"* has been introduced by DIRKEN who based it on his examination in a group of ± 100 normal individuals. He found that the oxygen percentage in 21 per cent O_2 respiration varied for the different test persons from 89½ per cent minimal to 98 per cent maximal. In 17 per cent oxygen respiration these figures were respectively 83 and 96 per cent. The extreme figures are hardly ever found; in most cases an oxygen content of 95—96 per cent is found in air respiration and of 92—93 per cent in 17 per cent oxygen respiration. DIRKEN has given the normal variation width of the oxygen percentage on a graduated scale (fig. 1). This scale has been divided into 10 equal parts. The figure 0 of the scale agrees, in 21 per cent respiration, with the lowest oxygen percentage ever found under these circumstances in normal persons (89½ per cent) while the figure 10 corresponds to the highest normal value (98 per cent). In 17 per cent oxygen respiration the figure 0 corresponds to an oxygen content of 84 per cent, figure 10 indicates an oxygen precentage of 96.



The figures 1—10 are denoted as *comparative figures. In normal individual this figure is about the same in 21 per cent and 17 per cent oxygen respiration,* viz. when in 21 per cent oxygen respiration an oxygen content of the blood of 95½ per cent is found (comparative figure 7) the oxygen content of the blood in 17 per cent oxygen respiration usually decreases to ± 92 per cent corresponding to the same comparative figure (7). However, there are also cases in which the decrease is more or less marked which means that a comparative figure 5—6, respectively 8—9 can also be considered as normal for the above mentioned example. A sharper fall (e.g. 7 to 44 or less) points to a disordered oxygen transport. The oxygen content of the blood in these cases, however, is still within the normal limits so that, in itself, it does not make the existence of a disordered diffusion probable. Only the mutual proportion of the two comparative figures (e.g. the abnormal sharp fall in 17 per cent oxygen respiration) points to this disorder.

For pathological cases it is necessary to lengthen the graduation downward. The comparative figures have here been indicated as —1, —2, —3 etc. When in 21 per cent oxygen respiration a negative comparative figure is found, this means that the oxygen content of the arterial blood is undoubtedly under normal.

It is to be expected that in these cases, in which, in air breathing, the disordered diffusion can be distinctly demonstrated, the comparative figure will drop sharply. This, however, is not always the case. Even the reverse is found (see discussion table III).

From the data of table I the following appears:

1. In air respiration the oxygen content of the arterial blood has not markedly decreased. No distinct pathological values (under 83 per cent) are found, not even in patients with extensive, often bilateral, tuberculous processes.

In group I and II the average oxygen content is identical to the content in normal individuals (95 per cent); in group III it is somewhat lower (93½ per cent). This finds expression in the rather low comparative figure 4—6 (the average comparative figure in normal individuals is 7—8).

2. In group I the average comparative figure, in inspiration of a gas mixture which contains 17 per cent oxygen, remains practically the same.

In group II a slight and in group III a somewhat more marked decrease of the comparative figure is found. The decrease in group II is so small that it does not give sufficient evidence for a disordered diffusion. The decrease in group III would not be of importance if it were found in one single case; however, as average value it is important. It points to slight disorders in the oxygen transport. This also becomes evident from the fact that in 10 of the 21 cases the fall of the comparative figure is more marked than in normal individuals (these cases are indicated in the table with a *).

Conclusion. In patients with not very extensive (group I) and rather extensive (group II) pulmonary tuberculosis the oxygen transport was not markedly disordered. In extensive bilateral affections (group III) latent disorders are present, which can be demonstrated by an artificial impediment of the diffusion through the lungs.

We must now ask ourselves how these facts can be explained. One feels inclined, as marked pathological changes in the lung-tissue are present, to expect important diffusion disorders. The fact that these disorders cannot be demonstrated can be explained only in two ways.

1. By accepting a normal function of the pathologically changed regions.
2. By supposing a total absence of oxygen diffusion in these regions.

A priori the first supposition is very improbable. Also the reduction of the lung voluminae (vital capacity) and of the ventilation reserves (respiratory level value) in these cases, strongly militates against a normal function of the pathological regions. This points to a lost function of the lungs in these regions.

However, when accepting a discontinuation of the oxygen diffusion, it remains to be asked why the oxygen content of the arterial blood has not markedly decreased in air respiration: the blood-flow through the pathological regions remains venous so that in the mixed blood (viz. the blood in the left heart) a decrease of the oxygen content had to occur.

This, however, will not be the case when the quantity of blood, flowing per time unit through the pathologically changed regions, diminished considerably. In the mixture-blood only a small quantity of non-arterialised blood is then present, so that no decrease of the oxygen content occurs.

The above mentioned data must, in all probability, be explained *by assuming that the ventilation* (and diffusion) *diminish simultaneously with the blood circulation* so that these regions are practically lost for the lung function. Only in this way can it be explained that the arterialisation of the blood remains normal in spite of the presence of important affections of the lung tissue. The blood flows only through the healthy lung parts while in the pathologically changed lung areas only a small quantity of blood circulates per time unit.

In spite of this adaptation of the body to pathological circumstances a disturbed gas transport is found in many cases of extensive pulmonary tuberculosis. This is due to a disordered function in the lung regions which show no tuberculous affection. It is known that in these regions an emphysema ultimately develops so that one is justified in associating it with the above mentioned disorders.

Following up our investigations mentioned in table I we made one and the same

examination *in a number of patients who were operated upon for collapse therapy*. The examinations were done with a view of investigating what influence the collapse condition of the lung had upon the arterialisation of the blood.

As, from the first investigation, it appeared that in patients suffering from not extensive or rather extensive tuberculous affections no marked disorders in the oxygen transport are present, the influence of the collapse therapy was only examined in these cases. Patients with extensive lung processes were excluded because here a secondary emphysema which might be present could have caused a disordered oxygen transport which would make an appreciation of the effect of the collapse therapy upon the oxygen transport impracticable.

The results of these observations are given in table II:

TABLE II.

Name	Group A				Name	Group B				Name	Group C			
	21 % of oxygen		17 % of oxygen			21 % of oxygen		17 % of oxygen			21 % of oxygen		17 % of oxygen	
	% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.		% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.		% O_2 art. bl.	Comp. fig.	% O_2 art. bl.	Comp. fig.
F. v. O.	98	10	95½	9.6	J. S.	95	6.5	90	5.4	A. S.	92	2.9	88½	4.3
J. H.	98½	10	96	10	M. G.	93	4.2	86½	2.7	Miss L.	95	6.5	91	6.2
J. W.	97	8.8	91½	6.6	C. v. T.	96	7.7	92½	7.4	J. N.	97	8.8	93½	8.1
Miss S.	97	8.8	92	6.9	H. G.	95½	7.1	92	7	F. d. G.	93½	4.7	88½	4.3
Miss M.	98	10	93	7.7	J. K.	96	7.7	92½	7.4	A. v. M.	*96½	8.3	88	3.8
J. V.	97½	8.8	93	7.7	v. B.	97	8.8	91	8.2	P. A.	94	5.3	88	3.8
P. S.	97	8.8	94½	8.9	J. B.	96½	8.3	93½	8.2	A. B.	94	5.3	91½	6.6
G. S.	96½	8.3	92	6.9	J. L.	*97	8.8	90½	5.8	F. v. Z.	92½	3.6	89	4.7
J. S.	*98	10	91½	6.6	R. L.	94½	5.9	90	5.4	A. G.	91	1.8	81½	—1.5
Miss A.D.	94	5.3	89½	5	A. M.	94½	5.9	91½	6.6	A. d. R.	*97½	9.6	88	3.8
P. v. L	*98	10	90½	5.8	M. v. R.	91½	2.3	88½	4.3	A. K.	93	4.2	88	3.8
Miss W.	97½	9.4	92	7	J. B.	*96½	8.3	82½	0.5	J. B.	*88	—2.0	78	—4.5
J. J.	95½	7.1	90	5.4	Miss G.V.	96½	8.3	91	6.2	J. S.	94	5.2	88	3.8
W. d. J.	96	7.7	90	5.4	Miss V.	96½	8.3	91½	6.6	C. W.	94½	5.8	88	3.8
Average	97%	8.8	92%	6.9	J. v. S.	93	4.2	89	4.7	J. R.	93½	4.7	88½	4.3
					J. S.	96	7.7	90	5.4	Miss v. N.	95½	7.1	90	5.4
					C. J.	95½	7.1	91	6.2	Average	94%	5 3	89%	4.6
					Miss L.	93	4.2	92	6.9					
					G. v. H.	94½	5.9	89½	5.0					
					Average	95%	6.4	90%	5.3					

The grouping is the same as in table I. Group A includes patients with small or rather extensive lung affections, who had a slight collapse of one lung (e.g. small pneumothorax, slight elevation of diaphragm after phrenico-exheresis, extrapleural pneumothorax). Group B: idem, but more marked collapse (large pneumothorax, marked elevation of diaphragm, plastic surgery of the apex) (resection of 4 or 5 ribs). Group C: idem, serious collapse, e.g. thoraxoplasty 7—8 ribs, maximal collapse of lung in pneumothorax, very highly elevated diaphragm after phrenico-exheresis, possibly combination of paralysis of diaphragm with thoracoplasty).

From the table it appears that:

1. In air respiration the oxygen content of the blood lies within the normal limits. There are no pathological values (except in one case of group C (J. B).)

2. The average oxygen content in the different groups is about the same. The average of group A is even somewhat higher than that found in normal individuals. The average of group B and C is identical to the normal values.

3. Except in some cases (from all three groups) the decrease of the comparative figure lies within the normal limits, in other words: as a rule no latent disorders in the oxygen transport can be demonstrated.

Conclusion:

From the data mentioned above it can be concluded that the collapse therapy, generally used in tuberculosis, has no unfavourable influence upon the arterialisation of the blood. Only in some cases slight disorders of the oxygen transport were found. With air respiration these were latent. These exceptional cases are present both with slight and with marked lung collapse.

The fact that a more or less severe collapse of the lung does not cause distinctly demonstrable disorders in the oxygen transport can only be understood when a marked diminution of the blood circulation in these regions — just as in the pathologically changed regions — is assumed. The reasons for this conception have already been discussed above. The collapsed lung fields, just as the pathologically changed regions, are practically lost for the function of the lungs.

In this way it can be explained that only a limitation of the lung function occurs (diminution of the respiration reserves) but that the gas metabolism in the lungs and the arterialisation of the blood remain normal as long as no secondary disorders in the remaining lung areas develop.

The secondary affections may be an emphysema (see above). It is also probable that — if there is a very marked diminution of the lung function — the vascular system in the functioning lung fields is finally overburdened so that in these cases it is very probable that a vascular congestion resp. edema develops. At the same time the restriction of the stream bed in the lung circulation generally causes a hypertrophy respectively a dilatation of the right heart.

On the so-called "short-circuit" in patients with pulmonary tuberculosis.

From the discussion of the results of the blood gas analysis of the cases of table I and II it appeared that, as a rule, no marked disturbances in the gas transport occur, except in those cases where a secondary emphysema developed.

The arterialisation of the mixed blood is practically normal which implies that the quantity of blood flowing through the pathologically changed respectively through the collapsed lung areas, is only small. In table III 10 cases are collected which form an exception to this rule.

TABLE III.

Name	21 % of oxygen	17 % of oxygen	Group
	% O_2 art. bl.	% O_2 art. bl.	
v. d. E.	92½	89	C
J. v. D.	87½	83½	111
J. H. E.	90½	89	C
N. H.	90½	89	C
v. G.	89	87	111
M. v. L.	93	92	C
C. S.	90	88	C
A. S.	92	88½	C
J. M.	90½	88	111 B
L. B.	92	91	C
Average	90.5	88.5	

In these cases the oxygen content of the blood in air respiration is rather low. In one case (J. v. D.) it is even pathological. The average oxygen content is 90.5, viz. about 5 per cent lower than normal. All these cases belong to group III, respectively to group C, in other words, extensive lung diseases or a pronounced lung collapse is present. In these cases, where a secondary emphysema is very probable, it would be expected that the decrease of the oxygen content is due to diffusion-disorders in the emphysematic lung areas.

This surmise, however, is not supported by the examination in a gas mixture which contains 17 per cent oxygen. If a disordered diffusion caused by the emphysema was present, the oxygen content in respiration of a 17 per cent oxygen mixture would fall more than normally. This now appears not to be the case; the decrease is only 2—3 per cent.

What can be the cause of this remarkable symptom? The possibility exists that a mistake has been made in the examination; this, however, is not probable as the technical examination was in all cases precisely the same. So there must be a special reason for this symptom. It could be explained if one assumed a rather important blood circulation in the pathological, respectively collapsed regions while the gas diffusion is removed. By BRAUER this condition is termed "Short-circuit". Some investigators think this short circuit to be of much importance in cases of tuberculosis. ZORN writes:

„Erstaunlich war es, dasz wir selbst bei kleineren aktiven spezifischen Prozessen „*stärkste* Insuffizienserscheinungen von Seiten der Lungen und teilweise des Kreislaufs „beobachten konnten.

„Auch war auffallend dasz bei benauester Beobachtung sehr häufig bei einigen Tuber- „kuloseformen Cyanose und Dyspnoe in Ruhe festgestellt werden, die eigentlich im „Widerspruch zu dem geringen röntgenologischen Befund standen. Eine Erklärung dieser „Tatsache wird von der BRAUERsche Schule darin gesucht, *dass ein Teil des Lungen-* „*gewebes* infolge der spezifischen Prozesse *sich nicht mehr aktiv an dem Sauerstoffaus-* „*tausch beteiligen kann, obschon eine Durchblutung dieser Bezirke noch stattfindet.* Dass „heiszt also: Das diese Bezirke durchfliessende Blut bleibt venös und wird in dieser Form „dem Herzen wieder zugeführt. Das Ergebnis für den Gesamtkreislauf ist dann eine unvoll- „ständige Arterialisierung des Blutes, was natürlich auf die Dauer zu Schädigungen der „Lungen- und Kreislauffunktion führen muss."

From the foregoing discussions (table I and II) it appears that this "short circuit" plays no or only a small part, as, even in very extensive lung affections, the oxygen content of the arterial blood is not pathological. Besides, the small fall, sometimes occurring in extensive processes, increases relatively in respiration of 17 per cent of oxygen. This would not be the case of the disorder in the oxygen transport was exclusively caused by the short circuit. In the cases of table III however, the reverse takes place. Here a decrease of the oxygen content in air respiration is present, only slightly increasing in inspiration of 17 per cent of oxygen. The difference in oxygen content of the blood in air- and in 17 per cent respiration is only 2—3 per cent.

An analogous difference is found in most normal individuals. The lungs of the patients mentioned in table III behave in this respect as normal lungs.

So the low oxygen content in air respiration cannot be attributed to a disordered diffusion in the non-tuberculous lung fields, nor can it be assumed that in the pathologically changed lung-areas the blood is partly supplied with oxygen as this would cause a relatively strong decrease of the oxygen content in 17 per cent oxygen respiration. This is not the case. Consequently we have to draw the conclusion that in these cases a rather large quantity of blood still flows through the pathologically changed (respectively collapsed) lung areas where it cannot be arterialised, so that a marked increase of the oxygen content in the mixed blood occurs, in other words a "short-circuit" develops.

This symptom, however, is but seldom found in pulmonary tuberculosis and when

present, is of no great importance as the decrease of the oxygen content of the blood caused by it, is too small to give clinical, respectively injurious symptoms.

SUMMARY.

The examination of the lung function by means of an analysis of the blood gas (determination of the oxygen content of the arterial blood) is discussed. For this we used the haemoxymeter of BRINKMAN by which the oxygen content can be determined exactly in small quantities of blood.

Beside the determination of the oxygen content of the blood in air respiration the oxygen content was also made in respiration of a gas mixture containing 17 per cent of oxygen (method of DIRKEN and KRAAN). In patient with pulmonary tuberculosis we found that:

1. The oxygen content of the arterial blood was normal in cases of small and rather extensive lung processes.
2. A small decrease of the oxygen content is present in several cases of extensive (bilateral) lung affections. In these cases the present disordered diffusion increases in respiration of a gas mixture, containing 17 per cent oxygen.
3. Collapse therapy generally does not cause disorders of the oxygenation of the blood. Even in cases of a very marked lung collapse the oxygen content of the blood is generally normal.
4. In some cases a slight diminution of the oxygen content in air respiration is present, which does not increase in respiration of a gas mixture which contains 17 per cent oxygen (so-called "short-circuit").

The following conclusions can now be drawn:

a. The oxygen transport (oxygen absorption in the blood) no longer takes place in the tuberculous, respectively collapsed lung areas.

b. The quantity of blood, flowing per time unit through these regions, has become strikingly smaller.

c. In extensive lung affections, respectively in marked collapse of the lung, a disordered diffusion in the remaining lung parts can develop (probably caused by a secondary emphysema).

d. In some cases of extensive lung diseases respectively in cases of a marked lung collapse a "short-circuit" is found, i.e. there is no longer a diffusion of oxygen in areas where moderate quantities of blood flow through these regions.

In contradistinction to other opinions it is urged that this "short-circuit" is only of little practical importance. It does not cause injurious effects as the diminution of the oxygen content of the blood is too small.

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